Unit 4-2-S: Magnetic Monopoles, Electromagnetic Angular Momentum, and the Quantization of Charge

Maxwell's equations, as we know and love them, are:

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi\rho, \qquad \boldsymbol{\nabla}\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}, \qquad \boldsymbol{\nabla}\cdot\mathbf{B} = 0, \qquad \boldsymbol{\nabla}\times\mathbf{B} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} \tag{4.2.S.1}$$

The equation $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ tells us there are no magnetic monopoles, i.e. no magnetic analog of a charge. Magnetic field lines have no sources or sinks; they either close upon themselves or come in from infinity and go out to infinity.

But why are there no magnetic monopoles? Simply because no one ever found one! Every so often people do experiments to look, and every so often people claim to have found a monopole. But those observations have never (as of yet) stood the test of time and reproducibly found monopoles. So to the best of our knowledge, there are none.

But that does not prevent people from thinking about what might be, if indeed monopoles were to exist. And such thought led Dirac to a clever argument that shows why charge would be quantized into integer multiples of a fundamental unit, were magnetic monopoles to exist. Recall, it is experimentally observed that electric charge is indeed quantized, but there is no apparent reason why it need be so. Dirac's argument provides a possible answer! Below we sketch out this argument.

Suppose there were magnetic monopoles g_i sitting at positions $\mathbf{r}_i$. We could then define a magnetic monopole density, just like we defined the electric charge density,

$$\eta(\mathbf{r}, t) = \sum_i g_i \delta(\mathbf{r} - \mathbf{r}_i(t)) \tag{4.2.S.2}$$

and then Maxwell's equation for $\boldsymbol{\nabla}\cdot\mathbf{B}$ would then be,

$$\boxed{\boldsymbol{\nabla}\cdot\mathbf{B} = 4\pi\eta} \tag{4.2.S.3}$$

We would also need to make a modification to Faraday's law, in the same spirit as Maxwell provided a correction to the magnetostatic Ampere's law.

$$\boldsymbol{\nabla}\times\mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} = 0 \quad \Rightarrow \quad \boldsymbol{\nabla}\cdot\left(\boldsymbol{\nabla}\times\mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}\right) = 0 \tag{4.2.S.4}$$

but

$$\boldsymbol{\nabla}\cdot\left(\boldsymbol{\nabla}\times\mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}\right) = \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{E}) + \frac{1}{c}\frac{\partial(\boldsymbol{\nabla}\cdot\mathbf{B})}{\partial t} = 0 + \frac{4\pi}{c}\frac{\partial\eta}{\partial t} \neq 0 \text{ in general} \tag{4.2.S.5}$$

In general, there should be a conservation of magnetic monopoles, just like there is a conservation of electric charges,

$$\frac{\partial\eta}{\partial t} = -\boldsymbol{\nabla}\cdot\mathbf{k} \tag{4.2.S.6}$$

where $\mathbf{k} = \sum_i g_i \mathbf{v}_i \delta(\mathbf{r} - \mathbf{r}_i)$ is the monopole current density, analogous to the electric current density $\mathbf{j}$.

One then has

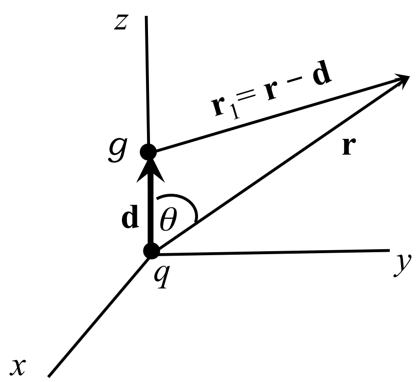
$$\boldsymbol{\nabla}\cdot\left(\boldsymbol{\nabla}\times\mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}\right) = -\frac{4\pi}{c}\boldsymbol{\nabla}\cdot\mathbf{k} \tag{4.2.S.7}$$

Then, in the spirit of Maxwell's correction to Ampere's law, we could say that if the divergence of the vectors on both sides are equal, then maybe the vectors themselves are equal, and so,

$$\boldsymbol{\nabla}\times\mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} = -\frac{4\pi}{c}\mathbf{k} \qquad \Rightarrow \qquad \boxed{\boldsymbol{\nabla}\times\mathbf{E} = -\frac{4\pi}{c}\mathbf{k} - \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}} \tag{4.2.S.8}$$

becomes the corrected Faraday's law. This corrected Faraday's law has the same form as the Maxwell corrected Ampere's law, except the terms on the right hand side enter with minus signs.

On to Dirac's argument!



Suppose one had an electric charge q sitting at the origin, and a magnetic monopole g sitting at position $d\hat{\mathbf{z}}$, a distance d above the charge on the $\hat{\mathbf{z}}$ axis.

The electric field from the electric charge at an observer at position $\mathbf{r}$ would be,

$$\mathbf{E}(\mathbf{r}) = q\,\frac{\hat{\mathbf{r}}}{r^2} = q\,\frac{\mathbf{r}}{r^3} \tag{4.2.S.9}$$

If we call $\mathbf{r}_1 = \mathbf{r} - \mathbf{d}$ the displacement from the magnetic monopole to the observer at $\mathbf{r}$, then the magnetic field at $\mathbf{r}$ would be,

$$\mathbf{B}(\mathbf{r}) = g\,\frac{\hat{\mathbf{r}}_1}{r_1^2} = g\,\frac{\mathbf{r}_1}{r_2^3} = g\,\frac{\mathbf{r} - \mathbf{d}}{|\mathbf{r} - \mathbf{d}|^3} \tag{4.2.S.10}$$

From Notes 4-2 we know that these electric and magnetic fields carry momentum, and so also angular momentum. The electromagnetic momentum density is,

$$\mathbf{\Pi} = \frac{1}{4\pi c}\mathbf{E} \times \mathbf{B} \tag{4.2.S.11}$$

and so the electromagnetic angular momentum density is,

$$\boldsymbol{\mathcal{L}} = \mathbf{r} \times \mathbf{\Pi} = \frac{\mathbf{r} \times (\mathbf{E} \times \mathbf{B})}{4\pi c} = \frac{qg}{4\pi c}\,\frac{\mathbf{r} \times [\mathbf{r} \times (\mathbf{r} - \mathbf{d})]}{r^3|\mathbf{r} - \mathbf{d}|^3} = -\frac{qg}{4\pi c}\,\frac{\mathbf{r} \times [\mathbf{r} \times \mathbf{d}]}{r^3|\mathbf{r} - \mathbf{d}|^3} \quad \text{since } \mathbf{r} \times \mathbf{r} = 0 \tag{4.2.S.12}$$

Now $\mathbf{r} \times \mathbf{d} = -rd\sin\theta\,\hat{\boldsymbol{\varphi}}$ and $\mathbf{r} \times [\mathbf{r} \times \mathbf{d}] = r^2 d\sin\theta\,\hat{\boldsymbol{\theta}}$. So $\boldsymbol{\mathcal{L}}(\mathbf{r})$ points in the $\hat{\boldsymbol{\theta}}$ direction. The $\hat{\boldsymbol{\theta}}$ vector has projections along the $\hat{\mathbf{z}}$ direction and in the xy plane. Due to the problem's rotational symmetry about the $\hat{\mathbf{z}}$ axis, when we integrate over all space to get the total angular momentum $\mathbf{L}$, the components in the xy plane will cancel out and vanish, and only the component along $\hat{\mathbf{z}}$ will remain non-zero. So we only need to compute this z-component.

$$\hat{\mathbf{z}} \cdot \mathbf{r} \times [\mathbf{r} \times \mathbf{d}] = \hat{\mathbf{z}} \cdot (\mathbf{r}[\mathbf{r} \cdot \mathbf{d}] - \mathbf{d}[\mathbf{r} \cdot \mathbf{r}]) = (\hat{\mathbf{z}} \cdot \mathbf{r})(\mathbf{r} \cdot \mathbf{d}) - (\hat{\mathbf{z}} \cdot \mathbf{d})(\mathbf{r} \cdot \mathbf{r}) = r^2 d\cos^2\theta - r^2 d \tag{4.2.S.13}$$

$$= r^2 d(\cos^2\theta - 1) = -r^2 d\sin^2\theta \tag{4.2.S.14}$$

where we used $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$.

We therefore have,

$$\hat{\mathbf{z}} \cdot \boldsymbol{\mathcal{L}} = \frac{qg}{4\pi c}\,\frac{r^2 d\sin^2\theta}{r^3|\mathbf{r} - \mathbf{d}|^3} = \frac{qg}{4\pi c}\,\frac{d\sin^2\theta}{r|\mathbf{r} - \mathbf{d}|^3} = \frac{qg}{4\pi c}\,\frac{d\sin^2\theta}{r(r^2 + d^2 - 2rd\cos\theta)^{3/2}} \tag{4.2.S.15}$$

Then we have for the total angular momentum of the electromagnetic fields,

$$\hat{\mathbf{z}} \cdot \mathbf{L} = \int d^3r\,\hat{\mathbf{z}} \cdot \boldsymbol{\mathcal{L}} = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\,\sin\theta \int_0^{\infty} dr\,r^2\,\frac{qg}{4\pi c}\,\frac{d\sin^2\theta}{r(r^2 + d^2 - 2rd\cos\theta)^{3/2}} \tag{4.2.S.16}$$

$$= \frac{qgd}{2c}\int_0^{\pi} d\theta\,\sin^3\theta \int_0^{\infty} dr\,\frac{r}{(r^2 + d^2 - 2rd\cos\theta)^{3/2}} \tag{4.2.S.17}$$

We can first do the r integration. Looking up in a table of integrals I find,

$$\int dx\,\frac{x}{(ax^2 + bx + c)^{3/2}} = \frac{2(bx + 2c)}{(b^2 - 4ac)\sqrt{ax^2 + bx + c}} \tag{4.2.S.18}$$

Apply this with $a = 1$, $b = -2d\cos\theta$, $c = d^2$ and $x = r$ to get

$$\int_0^\infty dr\, \frac{r}{(r^2 + d^2 - 2rd\cos\theta)^{3/2}} = \left[\frac{-4dr\cos\theta + 4d^2}{(4d^2\cos^2\theta - 4d^2)\sqrt{r^2 + d^2 - 2rd\cos\theta}}\right]_{r=0}^\infty \tag{4.2.S.19}$$

$$= \frac{-4d\cos\theta}{4d^2\cos^2\theta - 4d^2} - \frac{4d^2}{(4d^2\cos^2\theta - 4d^2)d} = \frac{1+\cos\theta}{d(1-\cos^2\theta)} = \frac{1+\cos\theta}{d(1+\cos\theta)(1-\cos\theta)} \tag{4.2.S.20}$$

$$= \frac{1}{d(1-\cos\theta)} \tag{4.2.S.21}$$

So

$$\hat{\mathbf{z}}\cdot\mathbf{L} = \frac{qgd}{2cd}\int_0^\pi d\theta\, \frac{\sin^3\theta}{1-\cos\theta} \tag{4.2.S.22}$$

Note, the distance d cancels in the numerator and the denominator, and so drops out of the answer. The total angular momentum is independent of the distance between the electric charge and the magnetic monopole.

To do the θ integration, make the substitutions $x = -\cos\theta$, $dx = d\theta\sin\theta$, and $\sin^2\theta = 1 - x^2$. We then get,

$$\hat{\mathbf{z}}\cdot\mathbf{L} = \frac{qg}{2c}\int_{-1}^1 dx\, \frac{1-x^2}{1+x} = \frac{qg}{2c}\int_{-1}^1 dx\,(1-x) = \frac{qg}{2c}\left[x - \frac{x^2}{2}\right]_{-1}^1 = \frac{qg}{c} \tag{4.2.S.23}$$

Now from quantum mechanics we know that angular momentum must be quantized in units of $\frac{\hbar}{2}$. We therefore must have,

$$\frac{qg}{c} = \frac{n\hbar}{2} \quad\Rightarrow\quad qg = \boxed{\frac{n\hbar c}{2} \qquad \text{where } n \text{ is an integer}} \tag{4.2.S.24}$$

The above would not in general hold if q and g could take any value along a continuum of values. The product qg can only be quantized in units of $\hbar c/2$ if the electric charge q is quantized in units of q_0 and the magnetic monopole is quantized in units of g_0, i.e. $q = mq_0$ and $g = m'g_0$ with m and m' integer, and then q_0g_0 is an integer multiple of $\hbar c/2$, say $q_0g_0 = n'\hbar c/2$ with n' integer. Then we have,

$$qg = mm'q_0g_0 = mm'n'\frac{\hbar c}{2} = \frac{n\hbar c}{2} \qquad \text{with } n = mm'n' \text{ integer} \tag{4.2.S.25}$$

So the existence of a magnetic monopole, together with the quantum mechanical quantization of angular momentum, would imply that electric charge must be quantized in integer multiples of some fundamental unit q_0. This argument would thus give us a means to understand why electric charge is indeed observed to be quantized.

Unfortunately, no one has ever convincingly demonstrated the existence of magnetic monopoles!