Unit 4-3-S: Forces and Torques on Electric and Magnetic Dipoles, and Interaction Energies

In these notes we consider the forces and torques on electric and magnetic dipoles in external electrostatic and magnetostatic fields, and also the electrostatic and magnetostatic interaction energies of the dipoles in an external field. By an *external* field, we mean the field created by sources other than the charges or currents that form the dipole.

Electric Dipoles - Force and Torque

Consider a localized charge distribution ρ with net charge $q = \int d^3r\, \rho = 0$.

The total force on ρ due to a slowly varying external electric field $\mathbf{E}$ is,

$$\mathbf{F} = \int d^3r\, \rho(\mathbf{r})\, \mathbf{E}(\mathbf{r}) \tag{4.3.S.1}$$

Define $\mathbf{r} = \mathbf{r}_0 + \mathbf{r}'$, where $\mathbf{r}_0$ is some fixed reference point in the center of the charge distribution ρ, and $\mathbf{r}'$ is the distance relative to $\mathbf{r}_0$. Then,

$$\mathbf{F} = \int d^3r'\, \rho(\mathbf{r}_0 + \mathbf{r}')\, \mathbf{E}(\mathbf{r}_0 + \mathbf{r}') \tag{4.3.S.2}$$

Since $\mathbf{E}$ is slowly varying on the length scale of the localized distribution ρ, we can expand in $\mathbf{r}'$,

$$\mathbf{F} = \int d^3r'\, \rho(\mathbf{r}_0 + \mathbf{r}')\, [\mathbf{E}(\mathbf{r}_0) + (\mathbf{r}' \cdot \boldsymbol{\nabla})\mathbf{E}(\mathbf{r}_0) + \dots] \tag{4.3.S.3}$$

$$= \left[\int d^3r \rho(\mathbf{r})\right] \mathbf{E}(\mathbf{r}_0) + \left[\int d^3r\, \rho(\mathbf{r})\mathbf{r}' \cdot \boldsymbol{\nabla}\right] \mathbf{E}(\mathbf{r}_0) \tag{4.3.S.4}$$

$$= (\mathbf{p} \cdot \boldsymbol{\nabla})\mathbf{E}(\mathbf{r}_0) \tag{4.3.S.5}$$

where the first term vanishes since the total charge $q = 0$, and $\mathbf{p}$ is the electric dipole moment computed about the point $\mathbf{r}_0$ (i.e. with $\mathbf{r}_0$ as the origin). However, since $q = 0$, we know the value of $\mathbf{p}$ is independent of the choice of the origin.

We thus have for the force on an electric dipole $\mathbf{p}$,

$$\boxed{\mathbf{F} = (\mathbf{p} \cdot \boldsymbol{\nabla})\mathbf{E} = \sum_{\alpha=1}^{3} p_\alpha \frac{\partial \mathbf{E}}{\partial r_\alpha}} \tag{4.3.S.6}$$

Note, for $\mathbf{E} = \text{constant}$, $\mathbf{F} = 0$.

The torque on $\mathbf{p}$ is given by,

$$\mathbf{N} = \int d^3r\, \rho(\mathbf{r})\, \mathbf{r} \times \mathbf{E}(\mathbf{r}) = \int d^3r\, \rho(\mathbf{r})\, \mathbf{r} \times [\mathbf{E}(\mathbf{r}_0) + \dots] \tag{4.3.S.7}$$

where again we expand $\mathbf{E}(\mathbf{r})$ about the point $\mathbf{r}_0$. To lowest order we then have

$$\boxed{\mathbf{N} = \mathbf{p} \times \mathbf{E}} \tag{4.3.S.8}$$

Magnetic Dipoles - Force and Torque

Consider a localized current distribution $\mathbf{j}(\mathbf{r})$.

The total force on $\mathbf{j}$ due to a slowly varying external magnetic field $\mathbf{B}$ is,

$$\mathbf{F} = \frac{1}{c}\int d^3r\,\mathbf{j}\times\mathbf{B} \tag{4.3.S.9}$$

Expand $\mathbf{B}$ about a point $\mathbf{r}_0$ in the center of the current distribution $\mathbf{j}$,

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}(\mathbf{r}_0) + (\mathbf{r}'\cdot\boldsymbol{\nabla})\mathbf{B}(\mathbf{r}_0) + \ldots \tag{4.3.S.10}$$

Then,

$$\mathbf{F} = \frac{1}{c}\left[\int d^3r\,\mathbf{j}(\mathbf{r})\right]\times\mathbf{B}(\mathbf{r}_0) + \frac{1}{c}\int d^3r\,\mathbf{j}(\mathbf{r})\times(\mathbf{r}'\cdot\boldsymbol{\nabla})\mathbf{B}(\mathbf{r}_0) + \ldots \tag{4.3.S.11}$$

For magnetostatics, we found in our discussion of the magnetic dipole approximation that $\int d^3r\,\mathbf{j} = 0$, so the first term above vanishes. Shifting the integration variable from $\mathbf{r}$ to $\mathbf{r}'$, the second term can be written as,

$$F_\alpha = \frac{\epsilon_{\alpha\beta\gamma}}{c}\int d^3r'\,j_\beta\,r'_\delta\,\partial_\delta B_\gamma \tag{4.3.S.12}$$

This involves the tensor that we also saw in our discussion of the magnetic dipole approximation,

$$\frac{1}{c}\int d^3r'\,j_\beta\,r'_\delta = -\frac{1}{c}\int d^3r'\,r'_\beta\,j_\delta = \frac{1}{2c}\int d^3r'\,\left[j_\beta\,r'_\delta - r'_\beta\,j_\delta\right] = -m_\sigma\epsilon_{\sigma\beta\delta} \tag{4.3.S.13}$$

where m_σ is the σ component of the magnetic dipole moment $\mathbf{m} = \frac{1}{2c}\int d^3r\,\mathbf{r}\times\mathbf{j}$.

Then,

$$F_\alpha = \epsilon_{\alpha\beta\gamma}\epsilon_{\sigma\beta\delta}(-m_\sigma)\partial_\delta B_\gamma = -(\delta_{\alpha\sigma}\delta_{\gamma\delta} - \delta_{\alpha\delta}\delta_{\sigma\gamma})m_\sigma\partial_\delta B_\gamma = \partial_\alpha(\mathbf{m}\cdot\mathbf{B}) - m_\alpha\boldsymbol{\nabla}\cdot\mathbf{B} \tag{4.3.S.14}$$

Since $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$, we have,

$$\boxed{\mathbf{F} = \boldsymbol{\nabla}(\mathbf{m}\cdot\mathbf{B})} \tag{4.3.S.15}$$

The torque on $\mathbf{m}$ is given by,

$$\mathbf{N} = \frac{1}{c}\int d^3r\,\mathbf{r}\times(\mathbf{j}\times\mathbf{B}) = \frac{1}{c}\int d^3r\,\left[\mathbf{j}\,(\mathbf{r}\cdot\mathbf{B}) - \mathbf{B}\,(\mathbf{r}\cdot\mathbf{j})\right] \tag{4.3.S.16}$$

Since $\mathbf{B}$ is slowly varying, to lowest order we can take $\mathbf{B}$ in the above to be the constant $\mathbf{B}(\mathbf{r}_0)$, and bring it outside the integral.

The second term in Eq. (4.3.S16) can be written as follows

$$\begin{aligned}
\int d^3r\,\mathbf{r}\cdot\mathbf{j} &= \int d^3r\,\mathbf{j}\cdot\boldsymbol{\nabla}\left(\frac{r^2}{2}\right) && \text{as } \boldsymbol{\nabla}\left(\frac{r^2}{2}\right) = \mathbf{r} \\
&= -\int d^3r\,(\boldsymbol{\nabla}\cdot\mathbf{j})\left(\frac{r^2}{2}\right) && \text{integrating by parts; the surface term} \to 0 \text{ as the current is localized} \\
&= 0 && \text{as } \boldsymbol{\nabla}\cdot\mathbf{j} = 0 \text{ in magnetostatics}
\end{aligned} \tag{4.3.S.17}$$

The first term in Eq. (4.3.S16) involves the tensor we have seen before in the magnetic dipole approximation,

$$\int d^3r\,\mathbf{j}\,\mathbf{r} = -\int d^3r\,\mathbf{r}\mathbf{j} = \frac{1}{2}\int d^3r\,[\mathbf{j}\,\mathbf{r} - \mathbf{r}\,\mathbf{j}] \tag{4.3.S.18}$$

So,

$$\mathbf{N} = \frac{1}{2c}\int d^3r\,\left[\mathbf{j}\,(\mathbf{r}\cdot\mathbf{B}) - \mathbf{r}\,(\mathbf{j}\cdot\mathbf{B})\right] = \frac{1}{2c}\int d^3r\,(\mathbf{r}\times\mathbf{j})\times\mathbf{B} \tag{4.3.S.19}$$

The integral in the last term above is just the magnetic dipole moment $\mathbf{m}$, so

$$\boxed{\mathbf{N} = \mathbf{m} \times \mathbf{B}} \tag{4.3.S.20}$$

Energy of an Electric Dipole in an External Field E

We have for the electrostatic energy,

$$\mathcal{E} = \frac{1}{8\pi} \int d^3r \, E^2 \tag{4.3.S.21}$$

Suppose the charge density ρ that produces $\mathbf{E}$ can be broken into two pieces, $\rho = \rho_1 + \rho_2$, with $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$, where $\boldsymbol{\nabla} \cdot \mathbf{E}_1 = 4\pi\rho_1$ and $\boldsymbol{\nabla} \cdot \mathbf{E}_2 = 4\pi\rho_2$. Then,

$$\mathcal{E} = \frac{1}{8\pi} \int d^3r \, \left[E_1^2 + E_2^2 + 2\mathbf{E}_1 \cdot \mathbf{E}_2\right] \tag{4.3.S.22}$$

Here the first term is the self energy of ρ_1, the second term is the self energy of ρ_2, and the third term is the interaction energy between ρ_1 and ρ_2.

Using similar arguments as we did earlier, the interaction energy piece can be written as,

$$\mathcal{E}_{\text{int}} = \frac{1}{4\pi} \int d^3r \, \mathbf{E}_1 \cdot \mathbf{E}_2 = \int d^3r \, \rho_1 \, \phi_2 = \int d^3r \, \rho_2 \, \phi_1 \tag{4.3.S.23}$$

where $\mathbf{E}_1 = -\boldsymbol{\nabla}\phi_1$ and $\mathbf{E}_2 = -\boldsymbol{\nabla}\phi_2$. Here the integrals are over all of space.

Let us apply the above to the interaction energy of an electric dipole in an external field $\mathbf{E}$. With ρ_1 the charge distribution of the dipole, and ϕ_2 the potential of the external electric field $\mathbf{E}$, we have

$$\mathcal{E}_{\text{int}} = \int d^3r \, \rho_1 \, \phi_2 \tag{4.3.S.24}$$

Assuming ϕ_2 varies slowly on the length scale of ρ_1, we can expand to linear order,

$$\phi_2(\mathbf{r}) = \phi_2(\mathbf{r}_0) + (\mathbf{r} - \mathbf{r}_0) \cdot \boldsymbol{\nabla}\phi_2(\mathbf{r}_0) \tag{4.3.S.25}$$

where $\mathbf{r}_0$ is the center of mass of (or any convenient reference point within) the distribution ρ_1. Then

$$\begin{aligned}
\mathcal{E}_{\text{int}} &= \int d^3r \, \rho_1(\mathbf{r}) \left[\phi_2(\mathbf{r}_0) + (\mathbf{r} - \mathbf{r}_0) \cdot \boldsymbol{\nabla}\phi_2(\mathbf{r}_0)\right] \\
&= q\phi_2(\mathbf{r}_0) + \left[\int d^3r \, \rho_1(\mathbf{r})(\mathbf{r} - \mathbf{r}_0)\right] \cdot \boldsymbol{\nabla}\phi_2(\mathbf{r}_0) \\
&= q\phi_2(\mathbf{r}_0) - \mathbf{p} \cdot \mathbf{E}
\end{aligned} \tag{4.3.S.26}$$

where q is the total charge in ρ_1, and $\mathbf{p}$ is the dipole moment or ρ_1 computed with respect to $\mathbf{r}_0$ as the origin.

For a *neutral* charge distribution with $q = 0$, and so with $\mathbf{p}$ independent of the choice of origin, the interaction energy of the dipole with $\mathbf{E}$ is given by,

$$\boxed{\mathcal{E}_{\text{int}} = -\mathbf{p} \cdot \mathbf{E}} \tag{4.3.S.27}$$

Note, $\mathcal{E}_{\text{int}}$ does *not* include the energy needed to make the dipole, nor the energy needed to make $\mathbf{E}$.

The interaction energy $\mathcal{E}_{\text{int}}$ is smallest when $\mathbf{p}$ is parallel to $\mathbf{E}$, so in a thermal ensemble, electric dipoles tend to align parallel to an applied $\mathbf{E}$.

Energy of a Magnetic Dipole in an External Field B

We found above that the force on a magnetic dipole $\mathbf{m}$ is

$$\mathbf{F} = \boldsymbol{\nabla}(\mathbf{m}\cdot\mathbf{B}) \tag{4.3.S.28}$$

If we regard this force as coming from the gradient of a potential energy U, then $\mathbf{F} = -\boldsymbol{\nabla}U$, and so we would expect,

$$\boxed{U = -\mathbf{m}\cdot\mathbf{B}} \tag{4.3.S.29}$$

as the energy of the dipole in the field $\mathbf{B}$. Equivalently, we can regard the energy of the dipole in the field to be the work one has to do to move the dipole into position from infinity; since the mechanical force we need to apply to move the dipole must be equal and opposite the the magnetic force above, one then has $W = -\int_\infty^{\mathbf{r}} \mathbf{F}\cdot \boldsymbol{d\ell} = -\int_\infty^{\mathbf{r}} \boldsymbol{\nabla}(\mathbf{m}\cdot\mathbf{B})\boldsymbol{d\ell} = -\mathbf{m}\cdot\mathbf{B}(\mathbf{r})$.

This is the correct energy to use in cases where $\mathbf{m}$ is due to the *intrinsic* magnetic moments of an atom or molecule, that arise from intrinsic electron or nuclear spin. In a thermal ensemble of such moments, the moments will thus tend to minimize U and align parallel to $\mathbf{B}$, giving a paramagnetic effect.

The answer comes out quite differently if we are talking about a magnetic moment produced by a classical current loop. To see this, consider what we would get if we tried to do the calculation in a similar way to how we computed the energy of an electric dipole in an electric field.

We have for the energy of a magnetostatic situation,

$$\mathcal{E} = \frac{1}{8\pi}\int d^3r\, B^2 \tag{4.3.S.30}$$

Suppose current $\mathbf{j}$ that produces $\mathbf{B}$ can be divided into two pieces, $\mathbf{j} = \mathbf{j}_1 + \mathbf{j}_2$, with $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2$, where $\boldsymbol{\nabla}\times\mathbf{B}_1 = \frac{4\pi}{c}\mathbf{j}_1$, and $\boldsymbol{\nabla}\times\mathbf{B}_2 = \frac{4\pi}{c}\mathbf{j}_2$. Then,

$$\mathcal{E} = \frac{1}{8\pi}\int d^3r\left[B_1^2 + B_2^2 + 2\mathbf{B}_1\cdot\mathbf{B}_2\right] \tag{4.3.S.31}$$

The first term is the self energy of $\mathbf{j}_1$, the second term is the self energy of $\mathbf{j}_2$, and the third term is the interaction energy between $\mathbf{j}_1$ and $\mathbf{j}_2$. We thus have,

$$\mathcal{E}_{\text{int}} = \frac{1}{4\pi}\int d^3r\,\mathbf{B}_1\cdot\mathbf{B}_2 = \frac{1}{c}\int d^3r\,\mathbf{j}_1\cdot\mathbf{A}_2 = \frac{1}{c}\int d^3r\,\mathbf{j}_2\cdot\mathbf{A}_1 \tag{4.3.S.32}$$

where $\mathbf{B}_1 = \boldsymbol{\nabla}\times\mathbf{A}_1$ and $\mathbf{B}_2 = \boldsymbol{\nabla}\times\mathbf{A}_2$, and the integrals are over all of space.

Now apply this $\mathcal{E}_{\text{int}}$ to the energy of a magnetic dipole created by the current $\mathbf{j}_1$ to an external magnetic field $\mathbf{B}$ given by $\mathbf{A}_2$. We have,

$$\mathcal{E}_{\text{int}} = \frac{1}{c}\int d^3r\,\mathbf{j}_1\cdot\mathbf{A}_2 = \frac{1}{c^2}\int d^3r\int d^3r'\,\frac{\mathbf{j}_1(\mathbf{r})\cdot\mathbf{j}_2(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{4.3.S.33}$$

where in the last step we used the solution, $\mathbf{A}_2(\mathbf{r}) = \dfrac{1}{c}\displaystyle\int d^3r'\,\frac{\mathbf{j}_2(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$, obtained for $\mathbf{A}_2$ in the Coulomb gauge.

Assume $\mathbf{A}_2$ varies slowly on the length scale of the localized $\mathbf{j}_1$, so that we can expand $\mathbf{A}_2(\mathbf{r})$ about some reference point $\mathbf{r}_0$ that lies somewhere in the middle of the localized $\mathbf{j}_1$. Then, to linear order in this expansion,

$$\mathcal{E}_{\text{int}} = \frac{1}{c}\int d^2r\,\mathbf{j}_1\cdot\mathbf{A}_2(\mathbf{r}_0) + \frac{1}{c}\int d^3r\sum_{k,l} j_{1k}(\mathbf{r}-\mathbf{r}_0)_l\partial_l A_k(\mathbf{r}_0) \tag{4.3.S.34}$$

From our discussion of the magnetic dipole approximation, we had $\int d^3r\,\mathbf{j} = 0$ in magnetostatics. Therefore the first term above vanishes, since $\mathbf{A}_2(\mathbf{r}_0)$ is a constant and may be taken outside the integral. We are left with,

$$\mathcal{E}_{\text{int}} = \left[\frac{1}{c}\int d^3r\, j_{1k}\, r_l\right]\partial_l A_k(\mathbf{r}_0) \qquad \text{where we use the summation convention with respect to } k \text{ and } l \tag{4.3.S.35}$$

Also from our discussion of the magnetic dipole approximation, we had

$$\int d^3r\, j_k\, r_l = -\int d^3r\, j_l\, r_k = \frac{1}{2}\int d^3r\, [j_k\, r_l - j_l\, r_k] = \frac{1}{2}\epsilon_{nkl}\int d^3r(\mathbf{j}\times\mathbf{r})_n = -\epsilon_{nkl}\, c\, m_n \tag{4.3.S.36}$$

where m_n is the nth component of the magnetic dipole moment of the current, $\mathbf{m} = \frac{1}{2c}\int d^3r\, \mathbf{r}\times\mathbf{j}$.

So we have,

$$\mathcal{E}_{\text{int}} = -m_n\, \epsilon_{nkl}\, \partial_l A_k = m_n\, \epsilon_{nlk}\, \partial_l A_k = \mathbf{m}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) = \mathbf{m}\cdot\mathbf{B} \tag{4.3.S.37}$$

This interaction energy is the same as what we had in Eq. (4.3.S29), except it has the *opposite* sign! If this were correct, then in a thermal ensemble, the current loops will want to minimize their energy and so would tend to align anti-parallel to $\mathbf{B}$, giving a diamagnetic effect!

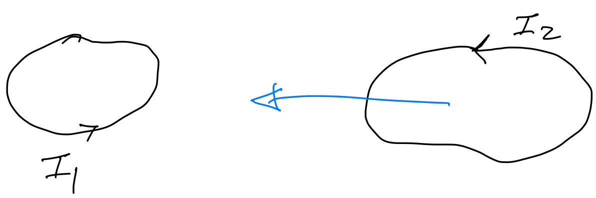
Why this difference??

1) When we integrate the work done against the magnetostatic force to move $\mathbf{m}$ into position from infinity, we found $U = -\mathbf{m}\cdot\mathbf{B}$.

2) When we compute the interaction energy from $\mathcal{E}_{\text{int}} = \frac{1}{c}\int d^3r\, \mathbf{j}_1\cdot\mathbf{A}_2 = \frac{1}{c^2}\int d^3r \int d^3r'\, \frac{\mathbf{j}_1(\mathbf{r})\cdot\mathbf{j}_2(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$ we found the energy $\mathcal{E}_{\text{int}} = +\mathbf{m}\cdot\mathbf{B}$.

To see which is correct, let us compute the interaction energy (2) directly, using the method of (1).

Consider two loops with currents I_1 and I_2. What is the work we need to do to quasistatically (i.e., slowly) move loop 2 in from infinity to its final position with respect to loop 1?



The magnetostatic force on loop 2 due to loop 1 is given by the Lorentz force,

$$\mathbf{F} = \frac{I_2}{c}\oint_2 d\boldsymbol{\ell}_2\times\mathbf{B}_1 \tag{4.3.S.38}$$

where the integral is around loop 2, and $\mathbf{B}_1$ is the magnetic field produced by loop 1.

Using the Biot-Savart law we can write for $\mathbf{B}_1$,

$$\mathbf{B}_1(\mathbf{r}) = \frac{I_1}{c}\oint_1 d\boldsymbol{\ell}_1\times\frac{(\mathbf{r}-\mathbf{r}_1)}{|\mathbf{r}-\mathbf{r}_1|^3} \qquad \text{where the integral goes over loop 1} \tag{4.3.S.39}$$

Therefore, substituting this into $\mathbf{F}$ we get

$$\mathbf{F} = \frac{I_1 I_2}{c^2}\oint_1\oint_2 d\boldsymbol{\ell}_2\times\frac{[d\boldsymbol{\ell}_1\times(\mathbf{r}_2-\mathbf{r}_1)]}{|\mathbf{r}_2-\mathbf{r}_1|^3} \tag{4.3.S.40}$$

Using the triple product rule we can write,

$$d\boldsymbol{\ell}_2\times[d\boldsymbol{\ell}_1\times(\mathbf{r}_2-\mathbf{r}_2)] = d\boldsymbol{\ell}_1[d\boldsymbol{\ell}_2\cdot(\mathbf{r}_2-\mathbf{r}_1)] - (\mathbf{r}_2-\mathbf{r}_1)(d\boldsymbol{\ell}_1\cdot d\boldsymbol{\ell}_2) \tag{4.3.S.41}$$

Substituting the first of these two terms into the expression for the force $\mathbf{F}$ above, we get

$$\oint_2 d\boldsymbol{\ell}_2\cdot\frac{(\mathbf{r}_2-\mathbf{r}_1)}{|\mathbf{r}_2-\mathbf{r}_1|^3} = -\oint_2 d\boldsymbol{\ell}_2\cdot\boldsymbol{\nabla}_2\left(\frac{1}{|\mathbf{r}_2-\mathbf{r}_1|}\right) = 0 \tag{4.3.S.42}$$

since the integral of a gradient around a closed loop always vanishes.

So, from the second term of the triple product rule we then get,

$$\mathbf{F} = -\frac{I_1 I_2}{c^2}\oint_1\oint_2 \boldsymbol{d\ell}_1\cdot\boldsymbol{d\ell}_2\frac{(\mathbf{r}_2-\mathbf{r}_1)}{|\mathbf{r}_2-\mathbf{r}_1|^3} \tag{4.3.S.43}$$

Now let's write $\mathbf{r}_2 = \mathbf{R}+\delta\mathbf{r}_2$ where $\mathbf{R}$ is the center of loop 2, and $\delta\mathbf{r}_2$ is the position of a segment of loop 2 relative to $\mathbf{R}$, and use

$$\frac{\mathbf{R}+\delta\mathbf{r}_2-\mathbf{r}_1}{|\mathbf{R}+\delta\mathbf{r}_2-\mathbf{r}_1|^3} = -\boldsymbol{\nabla}_{\mathbf{R}}\left(\frac{1}{|\mathbf{R}+\delta\mathbf{r}_2-\mathbf{r}_1|}\right) \qquad \text{where } \boldsymbol{\nabla}_{\mathbf{R}} \text{ differentiates with respect to } \mathbf{R} \tag{4.3.S.44}$$

We then get,

$$\mathbf{F} = \frac{I_1 I_2}{c^2}\oint_1\oint_2 \boldsymbol{d\ell}_1\cdot\boldsymbol{d\ell}_2\,\boldsymbol{\nabla}_{\mathbf{R}}\left(\frac{1}{|\mathbf{R}+\delta\mathbf{r}_2-\mathbf{r}_1|}\right) \tag{4.3.S.45}$$

Now to move the loop quasistatically, we need to apply a mechanical force that is equal but *opposite* the the magnetostatic force $\mathbf{F}$ above, $\mathbf{F}_{\text{mech}} = -\mathbf{F}$. Therefore the work we do in moving the loop 2 from infinity to its final position at $\mathbf{R}_0$ in,

$$W_{\text{mech}} = -\int_\infty^{\mathbf{R}_0}\mathbf{F}\cdot d\mathbf{R} = -\frac{I_1 I_2}{c^2}\oint_1\oint_2 \boldsymbol{d\ell}_1\cdot\boldsymbol{d\ell}_2\int_\infty^{\mathbf{R}_0} d\mathbf{R}\cdot\boldsymbol{\nabla}_{\mathbf{R}}\left(\frac{1}{|\mathbf{R}+\delta\mathbf{r}_2-\mathbf{r}_1|}\right) \tag{4.3.S.46}$$

$$= -\frac{I_1 I_2}{c^2}\oint_1\oint_2 \frac{\boldsymbol{d\ell}_1\cdot\boldsymbol{d\ell}_2}{|\mathbf{r}_2-\mathbf{r}_1|} \qquad \text{where we substituted back } \mathbf{r}_2 = \mathbf{R}_0+\delta\mathbf{r}_2 \tag{4.3.S.47}$$

$$= -M_{12}I_1I_2 \qquad \text{where } M_{12} \text{ is the mutual inductance of the two loops} \tag{4.3.S.48}$$

Or, for a more general current distribution, the above becomes

$$W_{\text{mech}} = -\frac{1}{c}\int d^3r_1\int d^3r_2\,\frac{\mathbf{j}_1(\mathbf{r}_1)\cdot\mathbf{j}_2(\mathbf{r}_2)}{|\mathbf{r}_2-\mathbf{r}_1|} = -\mathcal{E}_{\text{int}} \tag{4.3.S.49}$$

where $\mathcal{E}_{\text{int}}$ is the interaction energy of Eq. (4.3.S33).

Why the minus sign?? Why, in this calculation are we getting the negative of the energy we found from $\mathcal{E}_{\text{int}}$, or that we found from the inductance of current loops, $M_{12}I_1I_2$?

The minus sign we have here is the same minus sign we got when we found $U = -\mathbf{m}\cdot\mathbf{B}$ by integrating the force on the magnetic dipole.

Why don't we get $+\frac{1}{c}\int d^3r_1\int d^3r_2\,\frac{\mathbf{j}_1(\mathbf{r}_1)\cdot\mathbf{j}_2(\mathbf{r}_2)}{|\mathbf{r}_2-\mathbf{r}_1|}$, with the + sign that we expect from $\mathcal{E} = \frac{1}{8\pi}\int d^3r\,B^2$?

Answer: The answer is that we have left something out!

Faraday's Law: When we move loop 2, the magnetic flux through loop 2, from the field of loop 1, changes. This $d\Phi_2/dt$ creates an electromotive force emf $= \oint \boldsymbol{d\ell}\cdot\mathbf{E}$ around the loop that would cause the current in loop 2 to change. But in our calculation above, we have kept the current I_2 fixed. If we are to keep the current in loop 2 fixed at I_2, there must be a battery in the loop that does work to counter this induced emf. Similarly, the flux through loop 1, due to the field from loop 2, is changing, and so a battery in loop 1 must do work to keep the current in loop 1 fixed at I_1. We need to add to the above calculation of W_{mech} the work done by the batteries in loops 1 and 2 that keeps the currents constant.

The emf induced in loop 1 is $\mathcal{E}_1 = \oint_1 \boldsymbol{d\ell}_1\cdot\mathbf{E}_2$

The emf induced in loop 2 is $\mathcal{E}_2 = \oint_2 \boldsymbol{d\ell}_2\cdot\mathbf{E}_1$

In both cases we take the direction of integration around the loop to be in the direction of the current.

From Faraday's law, we have

$$\mathcal{E}_1 = -\frac{1}{c}\frac{d\Phi_1}{dt} \qquad \text{where } \Phi_1 \text{ is the magnetic flux through loop 1} \tag{4.3.S.50}$$

$$\mathcal{E}_2 = -\frac{1}{c}\frac{d\Phi_2}{dt} \qquad \text{where } \Phi_2 \text{ is the magnetic flux through loop 2} \tag{4.3.S.51}$$

To keep the current in the loops constant, the batteries need to provide an emf that *counters* these Faraday induced emfs. The work done by the batteries per unit time is therefore,

$$\frac{dW_{\text{battery}}}{dt} = -\mathcal{E}_1\, I_1 - \mathcal{E}_2\, I_2 \tag{4.3.S.52}$$

(the work to move a charge q around loop is $\mathcal{E}q$, so if I is the rate of charge flowing, the rate of work is $\mathcal{E}I$; we can also check the units: $\mathcal{E}I$ has units of [length$\cdot E$][q/s]=[length][force/s]=energy/s)

So then

$$\frac{dW_{\text{battery}}}{dt} = \frac{1}{c}\frac{d\Phi_1}{dt}\, I_1 + \frac{1}{c}\frac{d\Phi_2}{dt}\, I_2 \tag{4.3.S.53}$$

and

$$W_{\text{battery}} = \int_0^T dt\, \left[\frac{1}{c}\frac{d\Phi_1}{dt}\, I_1 + \frac{1}{c}\frac{d\Phi_2}{dt}\, I_2\right] \tag{4.3.S.54}$$

where $t = 0$ is when loop 2 is at infinity, and $t = T$ is when loop 2 is at its final position.

Since the currents I_1 and I_2 are kept constant as loop 2 moves, we can easily do the time integration to get,

$$W_{\text{battery}} = \frac{1}{c}\,\Phi_1\, I_1 + \frac{1}{c}\,\Phi_2\, I_2 \tag{4.3.S.55}$$

where Φ_1 and Φ_2 are the fluxes through the loops when loop 2 is in its final position, and we have assumed that the fluxes vanish when loop 2 is at infinity.

Since, by the definition of the mutual inductance, $\Phi_1 = cM_{12}I_2$, and $\Phi_2 = cM_{21}I_1 = cM_{12}I_1$ (since $M_{12} = M_{21}$), we finally have,

$$W_{\text{battery}} = 2M_{12}I_1I_2 \tag{4.3.S.56}$$

When we add this to the $W_{\text{mech}} = -M_{12}I_1I_2$ computed above, we then get,

$$W_{\text{total}} = W_{\text{mech}} + W_{\text{battery}} = -M_{12}I_1I_2 + 2M_{12}I_1I_2 = M_{12}I_1I_2 = +\frac{1}{c}\int d^3r_1 \int d^3r_2\, \frac{\mathbf{j}_1(\mathbf{r}_1)\cdot\mathbf{j}_2(\mathbf{r}_2)}{|\mathbf{r}_2 - \mathbf{r}_1|} \tag{4.3.S.57}$$

and so we get back the correct interaction energy $\mathcal{E}_{\text{int}}$ that came from $\frac{1}{4\pi}\int d^3r\, \mathbf{B}_1\cdot\mathbf{B}_2$.

Conclusion:

The magnetostatic interaction energy $\mathcal{E}_{\text{int}}$ includes the work done to maintain the currents constant as the current distributions move. Note that to compute $\mathcal{E}_{\text{int}}$ correctly, we had to invoke Faraday's law, and so even in this quasistatic process of slowly moving loop 2 into position, and computing the *magnetostatic* energy of the final configuration, we could not do that strictly in the context of magnetostatics – we need the dynamic effects of Faraday's law.

When we compute the interaction energy of a current loop dipole $\mathbf{m}$ and find $\mathcal{E}_{\text{int}} = \mathbf{m}\cdot\mathbf{B}$, this includes the energy needed to maintain the constant current producing $\mathbf{m}$.

When we integrated the force on the dipole to find the potential energy $U = -\mathbf{m}\cdot\mathbf{B}$, we regarded $\mathbf{m}$ as fixed, and so this *did not* include any contribution to the energy needed to maintain that dipole $\mathbf{m}$ at its fixed value. This is the correct energy expression to use when $\mathbf{m}$ comes from *intrinsic magnetic moments* due to the intrinsic spin of elementary particles such as the electron or nulceons – these cannot be viewed as arising from a current carrying loop!