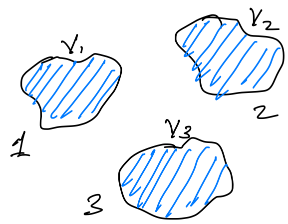
Unit 4-3: Capacitance and Inductance

The capacitance and inductance matrices are useful ways to compute the electrostatic and magnetostatic energies in static or quasistatic configurations

Capacitance

Consider a set of conductors with potentials $\phi(\mathbf{r}) = V_i$ fixed on conductor i. If the system is not enclosed in a box, we will also assume $\phi \to 0$ as $r \to \infty$.



From the uniqueness theorem we know that specifying V_i on each conductor is enough to uniquely determine the potential $\phi(\mathbf{r})$ everywhere in the system. We can therefore write this potential in the following form:

Let $\phi^{(i)}(\mathbf{r})$ be the solution to the boundary value problem

$$\nabla^2\phi^{(i)}(\mathbf{r}) = 0 \quad \text{and} \quad \phi^{(i)}(\mathbf{r}) = \begin{cases} 1 & \text{if } \mathbf{r} \text{ is on the surface of conductor } i \\ 0 & \text{if } \mathbf{r} \text{ is on the surface of any other conductor } j \neq i \end{cases} \tag{4.3.1}$$

Then, by superposition, the solution to the boundary value problem $\nabla^2\phi = 0$ with $\phi = V_i$ on conductor i is,

$$\phi(\mathbf{r}) = \sum_i V_i\, \phi^{(i)}(\mathbf{r}) \tag{4.3.2}$$

The surface charge density at $\mathbf{r}$ on the surface of conductor i is,

$$\sigma^{(i)}(\mathbf{r}) = -\frac{1}{4\pi}\frac{\partial\phi(\mathbf{r})}{\partial n} = -\frac{1}{4\pi}\sum_j V_j\, \frac{\partial\phi^{(j)}(\mathbf{r})}{\partial n} \tag{4.3.3}$$

where $\partial\phi/\partial n = (\boldsymbol{\nabla}\phi)\cdot\hat{\mathbf{n}}$ is the derivative in the direction normal to the surface at point $\mathbf{r}$.

The total charge on conductor i is then

$$Q_i = \int_{\mathcal{S}_i} da\, \sigma^{(i)}(\mathbf{r}) = -\frac{1}{4\pi}\sum_j V_j \int_{\mathcal{S}_i} da\, \frac{\partial\phi^{(j)}}{\partial n} \tag{4.3.4}$$

where $\mathcal{S}_i$ is the surface of conductor i. If we define the capacitance matrix,

$$C_{ij} \equiv -\frac{1}{4\pi}\int_{\mathcal{S}_i} da\, \frac{\partial\phi^{(j)}}{\partial n} \tag{4.3.5}$$

Then we have,

$$\boxed{Q_i = \sum_j C_{ij}\, V_j} \tag{4.3.6}$$

The charge on the conductor i is a linear function of the potentials V_j on all the conductors j. Note, the C_{ij} are determined solely by the geometry of the conductors, and not by the values of the potential or charge on the conductors.

Since we know that specifying the Q_i on each conductor will also uniquely determine $\phi(\mathbf{r})$, and hence the potentials on the conductors V_i, this implies that the capacitance matrix $\mathbf{C}$ is invertible, and

$$V_i = \sum_j \left[\mathbf{C}^{-1}\right]_{ij} Q_j \qquad \text{where } \mathbf{C}^{-1} \text{ is the inverse matrix of } \mathbf{C} \tag{4.3.7}$$

The electrostatic energy of the conductors is then

$$\mathcal{E} = \frac{1}{2}\int d^3r\, \rho\, \phi = \frac{1}{2}\sum_j Q_j\, V_j = \frac{1}{2}\sum_{i,j} C_{ij}\, V_i\, V_j = \frac{1}{2}\sum_{i,j} \left[\mathbf{C}^{-1}\right]_{ij} Q_i\, Q_j \tag{4.3.8}$$

We can compare this to the commonly defined capacitance of two conductors with equal and opposite charge. If conductor 1 has charge Q, and conductor 2 has charge $-Q$, and $V_1 - V_2$ is the potential difference from conductor 1 to conductor 2, then the usual definition for the capacitance C is,

$$C = \frac{Q}{V_i - V_2} \tag{4.3.9}$$

We can determine this C in terms of the elements of the 2×2 capacitance matrix C_{ij} defined above.

$$\left.\begin{aligned} Q &= C_{11}V_1 + C_{12}V_2 \\ -Q &= C_{21}V_1 + C_{22}V_2 \end{aligned}\right\} \quad \Rightarrow \quad V_2 = -\left(\frac{C_{11}+C_{21}}{C_{12}+C_{22}}\right) V_1 \tag{4.3.10}$$

Which gives

$$Q = \left[C_{11} - C_{12}\left(\frac{C_{11}+C_{21}}{C_{12}+C_{22}}\right)\right] V_1 \tag{4.3.11}$$

and

$$V_1 - V_2 = \left[1 + \left(\frac{C_{11}+C_{21}}{C_{12}+C_{22}}\right)\right] V_1 \tag{4.3.12}$$

so that

$$C = \frac{Q}{V_1 - V_2} = \frac{C_{11} - C_{12}\left(\dfrac{C_{11}+C_{21}}{C_{12}+C_{22}}\right)}{1 + \left(\dfrac{C_{11}+C_{21}}{C_{12}+C_{22}}\right)} \tag{4.3.13}$$

or

$$\boxed{C = \frac{C_{11}C_{22} - C_{12}C_{21}}{C_{11}+C_{12}+C_{21}+C_{22}}} \tag{4.3.14}$$

The capacitance can also be defined when the space between the conductors is filled with a dielectric material with dielectric constant ϵ. In this case, if Q_i is the free charge on conductor i, then Q_i/ϵ is the effective total charge on the conductor, to use in computing ϕ, since the charge Q_i is screened by the bound charge in the dielectric to give a net total charge of Q_i/ϵ on the conductor.

We then have,

$$\frac{Q_i}{\epsilon} = \sum_j C_{ij}^{(0)} V_j \tag{4.3.15}$$

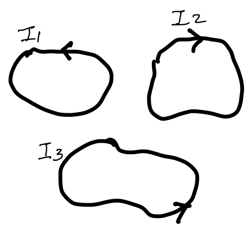
where $C_{ij}^{(0)}$ is the capacitance matrix appropriate to a vacuum between the conductors. This then gives,

$$Q_i = \sum_j \epsilon\, C_{ij}^{(0)}\, V_j = \sum_j C_{ij}\, V_j \tag{4.3.16}$$

where $C_{ij} \equiv \epsilon\, C_{ij}^{(0)}$ is the capacitance matrix with the dielectric filling. Adding the dielectric material between the conductors thus increases the capacitance by a factor of ϵ.

Inductance

Consider a set of current carrying loops C_i with currents I_i.



In the Coulomb gauge, we can write the magnetic vector potential $\mathbf{A}$ from these current loops as,

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c}\int d^3r'\,\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \sum_i \frac{I_i}{c}\oint_{C_i}\frac{d\boldsymbol{\ell}'}{|\mathbf{r}-\mathbf{r}'|} \tag{4.3.17}$$

where the integration variable $\mathbf{r}'$ goes over the current carrying loops C_i.

The magnetic flux through loop i is then,

$$\Phi_i = \int_{S_i} da\,\hat{\mathbf{n}}\cdot\mathbf{B} = \int_{S_i} da\,\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\times\mathbf{A} = \oint_{C_i} d\boldsymbol{\ell}\cdot\mathbf{A} \tag{4.3.18}$$

where S_i is the surface bounded by the loop C_i. We take the direction $\hat{\mathbf{n}}$ of the normal to S_i according to the direction we integrate around the loop, so as to be consistent with the right hand rule.

Substituting in the result of Eq. (4.3.17) for $\mathbf{A}$, we get,

$$\Phi_i = \sum_j \frac{I_j}{c}\oint_{C_i}\oint_{C_j}\frac{d\boldsymbol{\ell}\cdot d\boldsymbol{\ell}'}{|\mathbf{r}-\mathbf{r}'|} \tag{4.3.19}$$

Defining the mutual inductance matrix,

$$M_{ij} \equiv \frac{1}{c^2}\oint_{C_i}\oint_{C_j}\frac{d\boldsymbol{\ell}\cdot d\boldsymbol{\ell}'}{|\mathbf{r}-\mathbf{r}'|} \tag{4.3.20}$$

we then have,

$$\boxed{\Phi_i = c\sum_j M_{ij}I_j} \tag{4.3.21}$$

Note, the mutual inductance matrix M_{ij} is determined solely by the geometry of the current loops, and not by the values of the currents flowing in them. The inductance matrix is also symmetric, $M_{ij} = M_{ji}$.

The diagonal element $L_i \equiv M_{ii}$ is called the *self inductance* of loop i, and gives the magnetic flux through loop i that is due to the current flowing in the loop i itself.

The magnetostatic energy of the current loops can now be written in terms of the inductance matrix,

$$\mathcal{E} = \frac{1}{2c}\int d^3r\,\mathbf{j}\cdot\mathbf{A} = \frac{1}{2c}\sum_i I_i\oint_{C_i} d\boldsymbol{\ell}\cdot\mathbf{A} = \frac{1}{2c}\sum_i \Phi_i\,I_i = \frac{1}{2}\sum_{i,j} M_{ij}\,I_i\,I_j \tag{4.3.22}$$