Unit 4-4: Energy and Momentum of Electromagnetic Waves in a Vacuum

Electromagnetic Plane Waves in a Vacuum

In a vacuum, $\rho = 0$ and $\mathbf{j} = 0$. The microscopic Maxwell equations become,

$$\begin{aligned} &1)\quad \boldsymbol{\nabla}\cdot\mathbf{E} = 0 && 3)\quad \boldsymbol{\nabla}\cdot\mathbf{B} = 0 \\ &2)\quad \boldsymbol{\nabla}\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} && 4)\quad \boldsymbol{\nabla}\times\mathbf{B} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} \end{aligned} \tag{4.4.1}$$

Taking the curl of (2) gives,

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E}) - \nabla^2\mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{\nabla}\times\mathbf{B}) \tag{4.4.2}$$

Since $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$ by (1), we get

$$-\nabla^2\mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{\nabla}\times\mathbf{B}) = -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}\right) \qquad \text{where we used Faraday's law (4) in the last step} \tag{4.4.3}$$

So we have

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = 0 \qquad \text{this is the homogeneous wave equation with wave speed } c \tag{4.4.4}$$

Similarly, taking the curl of (4), and then using (3) and (2) gives

$$\nabla^2\mathbf{B} - \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} = 0 \tag{4.4.5}$$

Note, in MKS units the above wave equation would be $\nabla^2\mathbf{E} - \epsilon_0\mu_0\dfrac{\partial^2\mathbf{E}}{\partial t^2} = 0$, and the speed of the wave is $1/\sqrt{\epsilon_0\mu_0}$. It was the observation that the numerical value of $1/\sqrt{\epsilon_0\mu_0} = 3\times 10^8\, m/s$, as determined from the electromagnetically measured constants ϵ_0 and μ_0, was the same as the speed of light c, as measured in optical experiments, that led to the conclusion that light is an electromagnetic wave.

We consider a *simple harmonic plane wave* solution to the above wave equation, which has the form,

$$\begin{aligned} \mathbf{E}(\mathbf{r},t) &= \mathrm{Re}\left[\mathbf{E}_{\mathbf{k}}\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right] \\ \mathbf{B}(\mathbf{r},t) &= \mathrm{Re}\left[\mathbf{B}_{\mathbf{k}}\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right] \end{aligned} \tag{4.4.6}$$

Where "Re" means to take the real part of the complex valued expression. The physical fields $\mathbf{E}$ and $\mathbf{B}$ must always be real valued quantities, however we will usually find it convenient to use this complex exponential form to describe a plane wave. When we do that, we have to take the real part to get the physical fields. Often we will not bother to write "Re", but it will be understood as implied.

Here

$\mathbf{k}$	is the wavevector ($k = \lvert\mathbf{k}\rvert$ is the wave number)	
ω	is the angular frequency (or, for short, just the frequency)	
$\nu = \omega/2\pi$	is the frequency	
$T = 1/\nu$	is the period	(4.4.7)
$\lambda = 2\pi/\lvert\mathbf{k}\rvert$	is the wavelength	
$\lvert\mathbf{E}_{\mathbf{k}}\rvert, \lvert\mathbf{B}_{\mathbf{k}}\rvert$	are the amplitudes	

The fields are periodic in space in direction $\hat{\mathbf{k}}$, with a period λ, and they are periodic in time with a period T,

$$\mathbf{E}(\mathbf{r}+\lambda\hat{\mathbf{k}},t)=\mathbf{E}(\mathbf{r},t) \qquad \mathbf{E}(\mathbf{r},t+T)=\mathbf{E}(\mathbf{r},t) \tag{4.4.8}$$

The form above is called a *plane* wave because $\mathbf{E}(\mathbf{r},t)$ is constant in space on planes with normal $\hat{\mathbf{n}}\|\mathbf{k}$. This follows because if $\mathbf{r}_\perp$ is a displacement perpendicular to $\mathbf{k}$, then $\mathbf{r}_\perp\cdot\mathbf{k}=0$, and so $\mathbf{E}(\mathbf{r}+\mathbf{r}_\perp,t)=\mathbf{E}(\mathbf{r},t)$.

Properties of EM Plane Waves

The plane wave forms of Eq. (4.4.6) must solve Maxwell's equations. Requiring them to do so will give us the relations between the amplitudes $\mathbf{E}_\mathbf{k}$, $\mathbf{B}_\mathbf{k}$, the wavevector $\mathbf{k}$ and the frequency ω.

Using $\boldsymbol{\nabla}\cdot[\mathbf{E}_\mathbf{k}f(\mathbf{r})]=\mathbf{E}_\mathbf{k}\cdot\boldsymbol{\nabla}f(\mathbf{r})$ for a constant $\mathbf{E}_\mathbf{k}$, we have from (1),

$$\boldsymbol{\nabla}\cdot\mathbf{E}=0 \qquad \Rightarrow \qquad \mathrm{Re}\left[\mathbf{E}_\mathbf{k}\cdot\boldsymbol{\nabla}\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right]=\mathrm{Re}\left[i\mathbf{E}_\mathbf{k}\cdot\mathbf{k}\,\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right]=0 \qquad \Rightarrow \qquad \mathbf{E}_\mathbf{k}\cdot\mathbf{k}=0 \tag{4.4.9}$$

So the amplitude $\mathbf{E}_\mathbf{k}$ is perpendicular to the wavevector $\mathbf{k}$. Similarly, we have from (3),

$$\boldsymbol{\nabla}\cdot\mathbf{B}=0 \qquad \Rightarrow \qquad \mathbf{B}_\mathbf{k}\cdot\mathbf{k}=0 \tag{4.4.10}$$

and the amplitude $\mathbf{B}_\mathbf{k}$ is also perpendicular to the wavevector $\mathbf{k}$.

Now we use Ampere's law (4), and $\boldsymbol{\nabla}\times[\mathbf{B}_\mathbf{k}f(\mathbf{r})]=-\mathbf{B}_\mathbf{k}\times\boldsymbol{\nabla}f(\mathbf{r})$ for constant $\mathbf{B}_\mathbf{k}$, to get,

$$\boldsymbol{\nabla}\times\mathbf{B}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}=0 \tag{4.4.11}$$

$$\Rightarrow \quad \mathrm{Re}\left[\boldsymbol{\nabla}\times\mathbf{B}_\mathbf{k}\,\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}-\frac{1}{c}\mathbf{E}_\mathbf{k}\frac{\partial}{\partial t}\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right]=0 \tag{4.4.12}$$

$$\Rightarrow \quad \mathrm{Re}\left[-\mathbf{B}_\mathbf{k}\times\boldsymbol{\nabla}\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}+\frac{i\omega}{c}\mathbf{E}_\mathbf{k}\,\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right]=0 \tag{4.4.13}$$

$$\Rightarrow \quad \mathrm{Re}\left[i\mathbf{k}\times\mathbf{B}_\mathbf{k}\,\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}+\frac{i\omega}{c}\mathbf{E}_\mathbf{k}\,\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right]=0 \tag{4.4.14}$$

$$\Rightarrow \quad \mathbf{k}\times\mathbf{B}_\mathbf{k}+\frac{\omega}{c}\,\mathbf{E}_\mathbf{k}=0 \quad \Rightarrow \quad \mathbf{k}\times(\mathbf{k}\times\mathbf{B}_\mathbf{k})+\frac{\omega}{c}\,\mathbf{k}\times\mathbf{E}_\mathbf{k}=0 \tag{4.4.15}$$

Now since $\mathbf{k}\perp\mathbf{B}_\mathbf{k}$, then each $\mathbf{k}\times$ rotates $\mathbf{B}_\mathbf{k}$ by 90°, and so $\mathbf{k}\times(\mathbf{k}\times\mathbf{B}_\mathbf{k})=-k^2\mathbf{B}_\mathbf{k}$. We thus get

$$-k^2\mathbf{B}_\mathbf{k}=-\frac{\omega}{c}\mathbf{k}\times\mathbf{E}_\mathbf{k} \quad \Rightarrow \quad \mathbf{B}_\mathbf{k}=\frac{\omega}{ck^2}\,\mathbf{k}\times\mathbf{E}_\mathbf{k}=\frac{\omega}{ck}\,\hat{\mathbf{k}}\times\mathbf{E}_\mathbf{k} \tag{4.4.16}$$

Finally, from Eq. (4.4.4) we have

$$\nabla^2\mathbf{E}-\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}=0 \tag{4.4.17}$$

$$\Rightarrow \quad \mathrm{Re}\left[\mathbf{E}_\mathbf{k}\nabla^2\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}-\frac{1}{c^2}\mathbf{E}_\mathbf{k}\frac{\partial^2}{\partial t^2}\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right]=0 \tag{4.4.18}$$

$$\Rightarrow \quad \mathrm{Re}\left[\mathbf{E}_\mathbf{k}(-k^2)\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}+\frac{\omega^2}{c^2}\mathbf{E}_\mathbf{k}\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right]=0 \tag{4.4.19}$$

$$\Rightarrow \quad k^2=\frac{\omega^2}{c^2} \tag{4.4.20}$$

So, we get the *dispersion relation* for our EM plane waves,

$$\boxed{\omega = \pm ck} \tag{4.4.21}$$

Using this last result in Eq. (4.4.16) we then get,

$$\mathbf{B_k} = \pm\hat{\mathbf{k}} \times \mathbf{E_k} \qquad \text{so} \qquad |\mathbf{B_k}| = |\mathbf{E_k}| \tag{4.4.22}$$

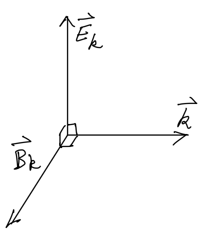
and the amplitudes of the electric and magnetic parts of the wave are equal.

By our definition, $k = |\mathbf{k}|$ is always positive. If we take the $(+)$ sign in Eq. (4.4.21), then ω is positive and the wave travels in the direction of $\mathbf{k}$. If we take the $(-)$ sign in Eq. (4.4.21), then ω is negative and the wave travels in the direction $-\mathbf{k}$. We can see this as follows. Take $\mathbf{k} = k\hat{\mathbf{z}}$. Then for $\mathbf{E} = \text{Re}[\mathbf{E_k} e^{i(kz-\omega t)}] = \mathbf{E_k}\cos(kz - \omega t)$, the maximum of the wave will be where the cosine is maximum, i.e., when $kz - \omega t = 0 \quad \Rightarrow \quad z = (\omega/k)t$. So when $\omega > 0$, the peak in the wave travels with speed $c = \omega/k$ in the $+\hat{\mathbf{z}}$ direction, i.e., in the direction of $\mathbf{k}$. When $\omega < 0$, the peak in the wave travels with speed $c = |\omega|/k$ in the $-\hat{\mathbf{z}}$ direction, i.e., in the direction $-\mathbf{k}$.

If we take $\tilde{\mathbf{k}}$ to have magnitude k and point in the direction of the wave propagation (so $\tilde{\mathbf{k}} = \mathbf{k}$ if $\omega > 0$, and $\tilde{\mathbf{k}} = -\mathbf{k}$ if $\omega < 0$), the $(\pm)$ sign in Eq. (4.4.22) ensures that $\tilde{\mathbf{k}}$, $\mathbf{E_k}$, and $\mathbf{B_k}$ form a right handed coordinate system.

In the rest of these notes, whenever we are dealing with a simple harmonic plane wave, we will always take the $(+)$ sign in Eq. (4.4.21), and the wave will be traveling in the direction of $\mathbf{k}$.

Summary



$$\left.\begin{array}{l}\mathbf{E_k} \perp \mathbf{k} \\ \mathbf{B_k} \perp \mathbf{k}\end{array}\right\} \quad \Rightarrow \quad \textit{transverse}\text{ polarization} \tag{4.4.23}$$

$$\mathbf{B_k} = \hat{\mathbf{k}} \times \mathbf{E_k} \quad \Rightarrow \quad \mathbf{B_k} \perp \mathbf{E_k} \quad \Rightarrow \quad \mathbf{k}, \mathbf{E_k}, \mathbf{B_k} \text{ form a right handed coordinate system} \tag{4.4.24}$$

$$\omega^2 = c^2k^2 \qquad \text{dispersion relation: how the frequency depends on the wavevector} \tag{4.4.25}$$

$$|\mathbf{B_k}| = |\mathbf{E_k}| \tag{4.4.26}$$

Since the Lorentz force on a charge is $\mathbf{F} = q\mathbf{E} + q\dfrac{\mathbf{v}}{c} \times \mathbf{B}$, the result $|\mathbf{B_k}| = |\mathbf{E_k}|$ means the force on a charge q from the magnetic field part of the electromagnetic wave is smaller than the force from the electric field part by a factor (v/c), and so can be ignored for a non-relativistically moving charge.

Most General Solution

Given that the plane waves of Eq. (4.4.6) are a solution to Maxwell's equations, and Maxwell's equations are linear, we can construct more general solutions by linear superposition,

$$\mathbf{E}(\mathbf{r}, t) = \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} \left[\mathbf{E}_{\mathbf{k},+}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \mathbf{E}_{\mathbf{k},-}\, e^{i(\mathbf{k}\cdot\mathbf{r}+\omega t)} \right] \tag{4.4.27}$$

where by definition we take $\omega = c|\mathbf{k}|$, and the two terms correspond to the two signs of the dispersion relation in Eq. (4.4.21).

Since any function $\mathbf{E}(\mathbf{r}, t)$ can be written as a Fourier transform, the above is also the most general wave solution.

Since the physical field $\mathbf{E}(\mathbf{r},t)$ must be a real valued quantity, we must have

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}^*(\mathbf{r},t) = \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} \left[\mathbf{E}^*_{\mathbf{k},+}\, \mathrm{e}^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \mathbf{E}^*_{\mathbf{k},-} \mathrm{e}^{-i(\mathbf{k}\cdot\mathbf{r}+\omega t)} \right] \qquad \text{now take } \mathbf{k} \leftrightarrow -\mathbf{k} \tag{4.4.28}$$

$$= \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} \left[\mathbf{E}^*_{-\mathbf{k},+}\, \mathrm{e}^{-i(-\mathbf{k}\cdot\mathbf{r}-\omega t)} + \mathbf{E}^*_{-\mathbf{k},-}\, \mathrm{e}^{-i(-\mathbf{k}\cdot\mathbf{r}+\omega t)} \right] \tag{4.4.29}$$

$$= \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} \left[\mathbf{E}^*_{-\mathbf{k},+}\, \mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}+\omega t)} + \mathbf{E}^*_{-\mathbf{k},-}\, \mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \right] \tag{4.4.30}$$

comparing with Eq. (4.4.27) we conclude,

$$\mathbf{E}^*_{-\mathbf{k},+} = \mathbf{E}_{\mathbf{k},-} \quad \text{and} \quad \mathbf{E}^*_{-\mathbf{k},-} = \mathbf{E}_{\mathbf{k},+} \quad \Rightarrow \qquad \mathbf{E}^*_{\mathbf{k},+} = \mathbf{E}_{-\mathbf{k},-} \quad \text{and} \quad \mathbf{E}^*_{\mathbf{k},-} = \mathbf{E}_{-\mathbf{k},+} \tag{4.4.31}$$

With $\omega = c|\mathbf{k}|$, and $\mathbf{k} = |\mathbf{k}|\hat{\mathbf{k}}$, we can write,

$$\mathbf{k}\cdot\mathbf{r} - \omega t = \mathbf{k}\cdot(\mathbf{r} - \mathbf{v}t) \qquad \text{where } \mathbf{v} = c\hat{\mathbf{k}} \text{ is the velocity of the wave} \tag{4.4.32}$$

while

$$\mathbf{k}\cdot\mathbf{r} + \omega t = \mathbf{k}\cdot(\mathbf{r} - \mathbf{v}t) \qquad \text{where } \mathbf{v} = -c\hat{\mathbf{k}} \text{ is the velocity of the wave} \tag{4.4.33}$$

If we only combine waves traveling with the same velocity $\mathbf{v}$, say with $\mathbf{v} = c\hat{\mathbf{z}}$, then we have,

$$\mathbf{E}(\mathbf{r},t) = \int_0^{\infty} \frac{dk}{(2\pi)}\, \mathbf{E}_{\mathbf{k},+}\, \mathrm{e}^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{v}t)} + \int_{-\infty}^{0} \frac{dk}{2\pi}\, \mathbf{E}_{\mathbf{k},-}\, \mathrm{e}^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{v}t)} = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \mathbf{E}_{\mathbf{k}}\, \mathrm{e}^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{v}t)} = \mathbf{E}(\mathbf{r}-\mathbf{v}t, 0) \tag{4.4.34}$$

where $\mathbf{k} = k\hat{\mathbf{z}} = |\mathbf{k}|\hat{\mathbf{k}}$, and $\mathbf{E}_{\mathbf{k}} = \mathbf{E}_{\mathbf{k},+}$ for $k > 0$ while $\mathbf{E}_{\mathbf{k}} = \mathbf{E}_{\mathbf{k},-}$ for $k < 0$.

Thus, if we know $\mathbf{E}$ at time $t = 0$, then we trivially know $\mathbf{E}$ at all other times t; we just translate the wave form at $t = 0$ by the spatial distance $\mathbf{v}t$ to get the form at time t. The wave form travels without changing its shape.

Energy and Momentum of a Simple Harmonic Plane Electromagnetic Wave

For a simple harmonic plane wave, if the amplitudes $\mathbf{E}_{\mathbf{k}}$ and $\mathbf{B}_{\mathbf{k}}$ are real values, then the physical fields are,

$$\begin{aligned} \mathbf{E}(\mathbf{r},t) &= \mathrm{Re}\left[\mathbf{E}_{\mathbf{k}}\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right] = \mathbf{E}_{\mathbf{k}}\cos(\mathbf{k}\cdot\mathbf{r}-\omega t) \\ \mathbf{B}(\mathbf{r},t) &= \mathrm{Re}\left[\mathbf{B}_{\mathbf{k}}\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right] = \hat{\mathbf{k}}\times\mathbf{E}_{\mathbf{k}}\cos(\mathbf{k}\cdot\mathbf{r}-\omega t) \end{aligned} \tag{4.4.35}$$

The energy density of the wave is then,

$$u(\mathbf{r},t) = \frac{1}{8\pi}\left[|\mathbf{E}(\mathbf{r},t)|^2 + |\mathbf{B}(\mathbf{r},t)|^2\right] = \frac{1}{8\pi}\left[|\mathbf{E}_{\mathbf{k}}|^2 + |\mathbf{E}_{\mathbf{k}}|^2\right]\cos^2(\mathbf{k}\cdot\mathbf{r}-\omega t) \tag{4.4.36}$$

$$= \frac{1}{4\pi}|\mathbf{E}_{\mathbf{k}}|^2\cos^2(\mathbf{k}\cdot\mathbf{r}-\omega t) \tag{4.4.37}$$

Note, it was crucial that we took the real valued form for $\mathbf{E}$ before squaring. This is always true when dealing with expressions using the complex exponential – one must take the real part before making any products. This is because, if z_1 and z_2 are two complex numbers, then $\mathrm{Re}[z_1 z_2] \neq \mathrm{Re}[z_1]\mathrm{Re}[z_2]$, and it is the latter that is the desired physical result.

The Poynting vector of the wave is,

$$\mathbf{S}(\mathbf{r},t) = \frac{c}{4\pi}\mathbf{E}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t) = \frac{c}{4\pi}\left[\mathbf{E}_{\mathbf{k}}\times(\hat{\mathbf{k}}\times\mathbf{E}_{\mathbf{k}})\right]\cos^2(\mathbf{k}\cdot\mathbf{r}-\omega t) \tag{4.4.38}$$

$$= \frac{c}{4\pi}\,\hat{\mathbf{k}}\,|\mathbf{E}_{\mathbf{k}}|^2\cos^2(\mathbf{k}\cdot\mathbf{r}-\omega t) = u(\mathbf{r},t)\,c\,\hat{\mathbf{k}} \tag{4.4.39}$$

so the energy current $\mathbf{S}$ is just the energy density u traveling with velocity $c\hat{\mathbf{k}}$.

The momentum density is

$$\boldsymbol{\Pi}(\mathbf{r},t) = \frac{1}{c^2}\mathbf{S}(\mathbf{r},t) = \frac{u(\mathbf{r},t)}{c}\,\hat{\mathbf{k}} \qquad (4.4.40)$$

so

$$u(\mathbf{r},t) = c\,|\boldsymbol{\Pi}(\mathbf{r},t)| \qquad (4.4.41)$$

which you might recognize as the energy-momentum relation of *photons!*

For visible light, the wavelength $\lambda \sim 5\times10^{-7}\,m \sim 5000\,\text{Å}$, and the period $T = \lambda/c \sim 1.6\times10^{-15}\,s$. For most classical experiments, the measurements take place on macroscopic scales, $\ell \gg \lambda$ and $t \gg T$.

Since the fields thus oscillate rapidly on the time scale of the measurement, we are therefore usually interested in time averaged quantities,

$$\langle u(\mathbf{r},t)\rangle = \frac{1}{T}\int_0^T dt\, u(\mathbf{r},t) = \frac{1}{8\pi}|\mathbf{E}_{\mathbf{k}}|^2 \qquad (4.4.42)$$

$$\langle \mathbf{S}(\mathbf{r},t)\rangle = c\langle u(\mathbf{r},t)\rangle\,\hat{\mathbf{k}} = \frac{c}{8\pi}|\mathbf{E}_{\mathbf{k}}|^2\,\hat{\mathbf{k}} \qquad (4.4.43)$$

$$\langle \boldsymbol{\Pi}(\mathbf{r},t)\rangle = \frac{1}{c}\,\langle u(\mathbf{r},t)\rangle = \frac{1}{8\pi c}|\mathbf{E}_{\mathbf{k}}|^2\,\hat{\mathbf{k}} \qquad (4.4.44)$$

These follow since $\frac{1}{2\pi}\int_0^{2\pi} d\Theta\,\cos^2\Theta = \frac{1}{2}$.

All the averages are constant in space.

The *intensity* I of light falling on a surface with unit normal $\hat{\mathbf{n}}$ is defined as the average power (energy/time) per area transported by the wave onto the surface,

$$I = \langle \mathbf{S}\rangle \cdot \hat{\mathbf{n}} \qquad (4.4.45)$$