Unit 5-3: EM Wave Propagation in Conductors

Conductors differ from dielectrics in that they have mobile conduction electrons which are not bound to atomic ion cores, but are free to move throughout the material. This means that we need to include them in the macroscopic charge density ρ and current density $\mathbf{j}$. The ion cores we will take as being fixed in place in the crystal lattice of the conductor. We will take the free charge and free current to be zero.

Frequency Dependent Conductivity

Similar to our simple model for a bound electron in a polarizable atom, we can write a simple model for a classical conduction electron in a metal. This is known as the "Drude Model." The equation of motion for the conduction electron is,

$$m\ddot{\mathbf{r}} = -e\mathbf{E}(t) - \frac{m}{\tau}\dot{\mathbf{r}} \tag{5.3.1}$$

Here m is the mass of the electron, $\mathbf{E}(t)$ is the time varying external electric field, and the last term on the right is a damping term, similar to the $-m\gamma\dot{\mathbf{r}}$ term we had for the bound electron. For historical reasons, we now call the damping constant $1/\tau$ instead of γ. Note the important point that the above equation of motion has no restoring force $-m\omega_0^2\mathbf{r}$ like we had for the bound electron, since the conduction electron is not bound to any ionic core.

Drude had in mind that τ, the *relaxation time*, was the average time between collisions of the electron with the ions in the crystal lattice, and the damping term modeled the random scattering of the electrons as they collided with the ions. But we now know from quantum mechanics that a particle in a periodic potential (such as the electric potential of the crystal lattice of ions) does not necessarily scatter randomly, but can propagate through due to coherent addition of the scattered waves off of each ion. For a perfectly periodic structure of ions, the Schrodinger equation will give stationary eigenstates for the electron that can carry a net momentum. We now know that the scattering of the conduction electrons is not due to collisions with the ions, but rather due to collisions with imperfections in the ion lattice – these could be either impurities, or due to thermal displacements of the ions from their perfect lattice positions; when one quantizes those thermal displacements, they are known as phonons.

For an oscillating external electric field, $\mathbf{E}(t) = \mathbf{E}_\omega \mathrm{e}^{-i\omega t}$ we will assume an oscillating response for the displacement of the conduction electron, $\mathbf{r}(t) = \mathbf{r}_\omega \mathrm{e}^{-i\omega t}$. Substituting into the equation of motion give,

$$\left(-\omega^2 - \frac{i\omega}{\tau}\right)\mathbf{r}_\omega = -\frac{e}{m}\mathbf{E}_\omega \qquad \Rightarrow \qquad \mathbf{r}_\omega = \frac{e}{m}\,\frac{1}{\omega^2 + i\omega/\tau}\,\mathbf{E}_\omega \tag{5.3.2}$$

The contribution of such electrons to the macroscopic current is

$$\mathbf{j}(t) = -en\dot{\mathbf{r}}(t) = \mathbf{j}_\omega \mathrm{e}^{-i\omega t} \qquad \Rightarrow \qquad \mathbf{j}_\omega = -en(-i\omega)\mathbf{r}_\omega \tag{5.3.3}$$

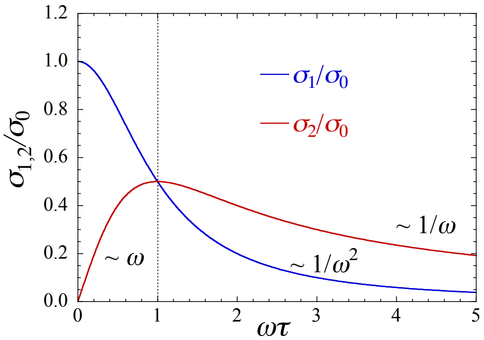
where n is the density of conduction electrons. Substituting in for $\mathbf{r}_\omega$ then gives,

$$\mathbf{j}_\omega = \frac{ne^2}{m}\,\frac{i\omega}{\omega^2 + i\omega/\tau}\,\mathbf{E}_\omega \;=\; \frac{ne^2\tau}{m}\,\frac{1}{1 - i\omega\tau}\,\mathbf{E}_\omega \tag{5.3.4}$$

This defines the frequency dependent conductivity $\sigma(\omega)$ for the conductor,

$$\mathbf{j}_\omega = \sigma(\omega)\mathbf{E}_\omega \qquad \Rightarrow \qquad \boxed{\sigma(\omega) = \frac{ne^2\tau}{m}\,\frac{1}{1 - i\omega\tau} = \frac{\sigma_0}{1 - i\omega\tau}} \tag{5.3.5}$$

where $\sigma_0 = \sigma(0) = \dfrac{ne^2\tau}{m}$ is the steady state ("dc") value of the conductivity.



The conductivity is a complex valued function of frequency with real and imaginary parts, $\sigma = \sigma_1 + i\sigma_2$,

$$\text{Re}\,\sigma \equiv \sigma_1 = \frac{\sigma_0}{1+\omega^2\tau^2} \tag{5.3.6}$$

and

$$\text{Im}\,\sigma \equiv \sigma_2 = \frac{\sigma_0\,\omega\tau}{1+\omega^2\tau^2} \tag{5.3.7}$$

The corresponding charge density ρ is obtained by applying the law of charge conservation. For a plane wave with $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_\omega e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$, we take similar forms for ρ and $\mathbf{j}$,

$$\rho = \rho_\omega e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \qquad \text{and} \quad \mathbf{j}(\mathbf{r},t) = \mathbf{j}_\omega e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{5.3.8}$$

Then

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla}\cdot\mathbf{j} \quad \Rightarrow \quad -i\omega\rho_\omega = -i\mathbf{k}\cdot\mathbf{j}_\omega \quad \Rightarrow \quad \rho_\omega = \frac{\mathbf{k}\cdot\mathbf{j}_\omega}{\omega} = \frac{\sigma(\omega)\mathbf{k}\cdot\mathbf{E}_\omega}{\omega} \tag{5.3.9}$$

Note that for a *transverse* polarized EM wave, where $\mathbf{k}\cdot\mathbf{E}_\omega = 0$, then $\rho_\omega = 0$.

Maxwell's Equations

We will assume a constant magnetic response, $\mu =$ constant, a frequency dependent conductivity $\sigma(\omega)$, and a frequency dependent dielectric function $\epsilon_b(\omega)$ describing the polarization of the bound (non-conduction) electrons that remain bound to the ionic cores.

Assuming all the fields have the plane wave form, $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_\omega e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$, we then have,

$$\mathbf{B}_\omega = \mu\mathbf{H}_\omega, \qquad \mathbf{D}_\omega = \epsilon_b(\omega)\mathbf{E}_\omega, \qquad \mathbf{j}_\omega = \sigma(\omega)\mathbf{E}_\omega, \qquad \rho_\omega = \frac{\sigma(\omega)}{\omega}\mathbf{k}\cdot\mathbf{E}_\omega \tag{5.3.10}$$

Maxwell's equations then become

$$1)\quad \boldsymbol{\nabla}\cdot\mathbf{D} = 4\pi\rho \quad\Rightarrow\quad i\mathbf{k}\cdot\mathbf{D}_\omega = i\mathbf{k}\cdot\epsilon_b\mathbf{E}_\omega = 4\pi\rho_\omega = \frac{4\pi\sigma}{\omega}\mathbf{k}\cdot\mathbf{E}_\omega \quad\Rightarrow\quad i\mathbf{k}\cdot\mathbf{E}_\omega\left(\epsilon_b + \frac{4\pi i\sigma}{\omega}\right) = 0 \tag{5.3.11}$$

$$2)\quad \boldsymbol{\nabla}\cdot\mathbf{B} = 0 \quad\Rightarrow\quad i\mu\mathbf{k}\cdot\mathbf{H}_\omega = 0 \tag{5.3.12}$$

$$3)\quad \boldsymbol{\nabla}\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} \quad\Rightarrow\quad i\mathbf{k}\times\mathbf{E}_\omega = \frac{i\omega}{c}\mathbf{B}_\omega = \frac{i\omega\mu}{c}\mathbf{H}_\omega \tag{5.3.13}$$

$$4)\quad \boldsymbol{\nabla}\times\mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t} \quad\Rightarrow\quad i\mathbf{k}\times\mathbf{H}_\omega = \frac{4\pi}{c}\mathbf{j}_\omega - \frac{i\omega}{c}\mathbf{D}_\omega = \frac{4\pi}{c}\sigma\mathbf{E}_\omega - \frac{i\omega}{c}\epsilon_b\mathbf{E}_\omega \tag{5.3.14}$$

$$\Rightarrow\quad i\mathbf{k}\times\mathbf{H}_\omega = \frac{-i\omega}{c}\left(\epsilon_b + \frac{4\pi i\sigma}{\omega}\right)\mathbf{E}_\omega \tag{5.3.15}$$

Note: all the above equations (1) – (4) look *exactly* like what we had for the dielectric material, provided we define

$$\boxed{\epsilon(\omega) = \epsilon_b(\omega) + \frac{4\pi i\sigma(\omega)}{\omega}} \tag{5.3.16}$$

So all our results for the dielectric case carry over to conductors, provided we make the above definition of $\epsilon(\omega)$, so as to include the effects of both bound electrons (via ϵ_b) and free mobile conduction electrons (via σ).

Discussion Question 5.3

Suppose that, instead of thinking of the conduction electrons as being "free" charges and so giving rise to a macroscopic $\mathbf{j}$, we treated them the same way we treated bound electrons, and computed their electric dipole moment $\mathbf{p}_\omega = -e\mathbf{r}_\omega$, with $\mathbf{r}_\omega$ as above. We could then define the polarizability $\alpha_c(\omega)$ for a conduction electron by $\mathbf{p}_\omega = \alpha_c(\omega)\mathbf{E}_\omega$. The total electric susceptibility of the material would then be $\chi_e = n_b\alpha_b + n\alpha_c$, where n_b is the density of the bound (non-conduction) electrons and α_b their atomic polarizability, and n is the density of the conduction electrons. The dielectric function of the material would then be $\epsilon(\omega) = 1 + 4\pi\chi_e(\omega)$. Show that this approach gives the same result as given by Eq. (5.3.16).

In particular, the dispersion relation for *transverse* waves in the conductor is

$$\boxed{k^2 = \frac{\omega^2}{c^2}\mu\epsilon(\omega)} \tag{5.3.17}$$

The main difference between wave propagation in dielectrics vs conductors has to do with the contribution that the term $4\pi i\sigma/\omega$ makes to the real and imaginary parts of $\epsilon(\omega)$.

Recall, for our simple Drude model, $\sigma(\omega) = \dfrac{\sigma_0}{1 - i\omega\tau}$ with $\sigma_0 = \dfrac{ne^2\tau}{m}$.

We now consider different regions of behavior.

Low frequencies: $\omega \ll 1/\tau$ and $\omega \ll \omega_0$ where ω_0 is the resonant frequency of the bound electrons.

In this limit,

$\epsilon_b(\omega) \approx \epsilon_b(0)$, a real valued constant.

$\sigma(\omega) \approx \sigma_0$, a real valued constant.

$$\Rightarrow \qquad \boxed{\epsilon(\omega) \approx \epsilon_b(0) + \frac{4\pi i\sigma_0}{\omega}} \tag{5.3.18}$$

The contribution from the conductivity term will give a large contribution to the imaginary part of ϵ at low frequencies as $\omega \to 0$. This gives rise to strong dissipation at low frequencies.

We will write $\epsilon = \epsilon_1 + i\epsilon_2$, where $\epsilon_1 = \mathrm{Re}\,[\epsilon]$ and $\epsilon_2 = \mathrm{Im}\,[\epsilon]$ are real valued.

When

$$\frac{\epsilon_2}{\epsilon_1} = \frac{4\pi\sigma_0}{\omega\epsilon_b(0)} \gg 1 \quad \text{we call this regime a } \textit{good} \text{ conductor.} \tag{5.3.19}$$

The conduction electrons dominate the response. This is similar to the region of *resonant absorption* of a dielectric. Waves are strongly attenuated.

When

$$\frac{\epsilon_2}{\epsilon_1} = \frac{4\pi\sigma_0}{\omega\epsilon_b(0)} \ll 1 \quad \text{we call this regime a } \textit{poor} \text{ conductor.} \tag{5.3.20}$$

There is little absorption of energy by the conduction electrons and waves propagate with little attenuation.

Because of the factor of ω in denominator, one always enters the good conductor region when ω gets sufficiently small.

From the dispersion relation, $k = \frac{\omega}{c}\sqrt{\mu\epsilon}$, we have the following.

<u>for a good conductor</u> where $\epsilon_2 \gg \epsilon_1$,

$$\epsilon \approx i\epsilon_2 = \frac{4\pi i\sigma_0}{\omega} \quad \Rightarrow \quad k = k_1 + ik_2 = \frac{\omega}{c}\sqrt{\frac{4\pi\mu\sigma_0}{\omega}i} \tag{5.3.21}$$

Since $\sqrt{i} = (1+i)/\sqrt{2}$ we have,

$$k_1 = k_2 = \frac{\omega}{c}\sqrt{\frac{4\pi\mu\sigma_0}{2\omega}} = \frac{1}{c}\sqrt{2\pi\mu\sigma_0\omega} \tag{5.3.22}$$

Since the field is $\mathbf{E} = \mathbf{E}_\omega e^{i(kz-\omega t)} = \mathbf{E}_\omega e^{-k_2 z} e^{i(k_1 z - \omega t)}$ one defines,

$$\boxed{\delta \equiv \frac{1}{k_2} = \frac{c}{\sqrt{2\pi\mu\sigma_0\omega}} \sim \frac{1}{\sqrt{\omega}}} \qquad \text{the } \textit{skin depth} \tag{5.3.23}$$

The skin depth determines the distance the wave can propagate into the conductor before the amplitude significantly decays. Since $\delta \sim 1/\sqrt{\omega}$, the skin depth increases as the frequency decreases.

The phase shift between the oscillations of $\mathbf{E}$ and $\mathbf{H}$ is

$$\phi = \arctan(k_2/k_1) \approx \arctan(1) = 45^\circ \tag{5.3.24}$$

Since $|k| = \sqrt{k_1^2 + k_2^2} = \sqrt{2}k_1$, the amplitude ratio is

$$\frac{|\mathbf{H}_\omega|}{|\mathbf{E}_\omega|} = \frac{c|k|}{\omega\mu} = \frac{\sqrt{2}ck_1}{\omega\mu} = \frac{\sqrt{2}c}{\omega\mu}\frac{1}{c}\sqrt{2\pi\mu\sigma_0\omega} = \sqrt{\frac{4\pi\sigma_0}{\omega\mu}} \sim \frac{1}{\sqrt{\omega}} \tag{5.3.25}$$

As $\omega \to 0$, most of the energy of the wave is carried by the magnetic field part.

High frequencies: $\quad \omega \gg 1/\tau$ and $\omega \gg \omega_0$

In this limit

$$\epsilon_b(\omega) \approx 1 \tag{5.3.26}$$

$$\sigma(\omega) \approx \frac{\sigma_0}{-i\omega\tau} = \frac{ine^2\tau}{m\omega\tau} = \frac{ine^2}{m\omega} \qquad \text{using } \sigma_0 = \frac{ne^2\tau}{m} \tag{5.3.27}$$

Note, σ is now pure imaginary and independent of τ. That it is independent of τ can be understood, because when $\omega \gg 1/\tau$, the electric field has many oscillations in between two successive collisions of the conduction electron, so the collisions are not effecting the response.

Then,

$$\epsilon(\omega) = \epsilon_b + \frac{4\pi i\sigma}{\omega} \approx 1 - \frac{4\pi ne^2}{m\omega^2} \tag{5.3.28}$$

If we define $\omega_p = \sqrt{\frac{4\pi ne^2}{m}}$ as the plasma frequency of the conduction electrons (n is the density of the conduction electrons), then we have,

$$\boxed{\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}} \qquad \text{and } \epsilon(\omega) \text{ is real valued.} \tag{5.3.29}$$

There are now two cases to consider. Considering the dispersion relation $k = (\omega/c)\sqrt{\mu\epsilon}$ we have,

1) If $\omega > \omega_p$ then $\epsilon > 0$. Then k is purely real valued, and so this is then a region of transparent propagation.

$$k = k_1 = \frac{\omega}{c}\sqrt{\mu\epsilon} \qquad \text{and} \qquad k_2 = 0 \tag{5.3.30}$$

2) If $\omega < \omega_p$ then $\epsilon < 0$. Then k is purely imaginary, and so this is then a region of total reflection!

$$k = ik_2 \quad \text{where} \quad k_2 = \frac{\omega}{c}\sqrt{\mu|\epsilon|} \qquad \text{and} \qquad k_1 = 0 \tag{5.3.31}$$

The plasma frequency ω_p thus gives the crossover frequency where one transitions from a region of total reflection to a region of transparent propagation.

For typical metals $\tau \sim 10^{-14}\,\text{sec}$, $\omega_p \sim 10^{16}\,\text{sec}^{-1}$, $\lambda_p = \dfrac{2\pi c}{\omega_p} \sim 3 \times 10^3\text{Å}$, which is just short of the visible spectrum where $\lambda \sim 5 \times 10^3\text{Å}$. The crossover from reflecting to transparent thus occurs at frequencies somewhat higher than the visible range.

Example

There is one effect of ω_p that you may have experienced for yourself. You may have notice when you listen to the radio (the old fashioned way with an antenna, not online!) that you can usually only pick up local FM stations, while (particularly at night) you can sometimes pick up AM stations from far away. The reason is the plasma frequency!

The earth is surrounded by the ionosphere, which is a layer of charged gas. In many respects the charged particles in the ionosphere behave like the conduction electrons in a metal. The plasma frequency of the ionosphere is such that for AM radio frequencies, $\omega_{\text{AM}} < \omega_p$, but for FM radio frequencies, $\omega_{\text{FM}} > \omega_p$. When the antenna of the radio station broadcasts its signal, the ionosphere is transparent to the FM signal which passes through and goes out into space. However, the ionosphere is reflective to the AM signal, which then reflects back to earth. At night the ionosphere moves in closer to the earth and the reflection is enhanced. When you hear in Rochester an AM station from Chicago or Boston, you are hearing the signal reflected from the ionosphere.

Longitudinal Modes

The above discussion was for transverse EM waves in a conductor, where **E** and **H** are perpendicular to **k**. Here we want to ask if Maxwell's equations allow for any longitudinal modes, where **E** or **H** can be parallel to **k**.

Consider first the magnetic field.

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \quad \Rightarrow \quad i\mu\mathbf{k}\cdot\mathbf{H}_\omega = 0 \quad \Rightarrow \quad \mathbf{H}_\omega \perp \mathbf{k} \text{ is transverse, } \textit{or } \mathbf{k} = 0 \tag{5.3.32}$$

The case $\mathbf{k} = 0$ corresponds to a spatially uniform **H**. Now if $\mathbf{k} = 0$, then Faraday's law gives,

$$i\mathbf{k}\times\mathbf{E}_\omega = \frac{i\omega\mu}{c}\mathbf{H}_\omega = 0 \quad \Rightarrow \quad \omega = 0 \qquad \text{if we are to have a non zero } \mathbf{H}_\omega \tag{5.3.33}$$

So the only possible longitudinal **H** is a spatially uniform, constant in time, magnetic field.

Now let's consider the electric field.

Using Eq. (5.3.11) we have,

$$\boldsymbol{\nabla}\cdot\mathbf{D} = 4\pi\rho \quad \Rightarrow \quad i\epsilon(\omega)\mathbf{k}\cdot\mathbf{E}_\omega = 0 \quad \Rightarrow \quad \mathbf{E}_\omega \perp \mathbf{k} \text{ is transverse, } \textit{or } \epsilon(\omega) = 0 \tag{5.3.34}$$

So if $\mathbf{E}_\omega \| \mathbf{k}$ but $\epsilon(\omega) = 0$, then we can satisfy all the other Maxwell's equations, as we see below.

$$\text{Faraday:} \quad i\mathbf{k}\times\mathbf{E}_\omega = \frac{i\omega\mu}{c}\mathbf{H}_\omega = 0 \quad \Rightarrow \quad \text{is satisfied if } \mathbf{H}_\omega = 0 \tag{5.3.35}$$

$$\text{Ampere:}\quad i\mathbf{k}\times\mathbf{H}_\omega = \frac{-i\omega\epsilon(\omega)}{c}\mathbf{E}_\omega \qquad \text{is satisfied since } \mathbf{H}_\omega = 0 \text{ and } \epsilon(\omega) = 0 \tag{5.3.36}$$

So we can have a longitudinal electric field at frequencies such that $\epsilon(\omega) = 0$.

At low frequencies, $\omega \ll 1/\tau$ and $\omega \ll \omega_0$, we had

$$\epsilon(\omega) = \epsilon_b(0) + \frac{4\pi i\sigma_0}{\omega} \qquad \text{so } \epsilon = 0 \text{ when} \qquad \omega = -\frac{4\pi i\sigma_0}{\epsilon_b(0)} \tag{5.3.37}$$

Since $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_\omega \mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ then $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_\omega \mathrm{e}^{-\frac{4\pi\sigma_0}{\epsilon_b(0)}t}\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}}$

So if we set up a longitudinal $\mathbf{E}$ field in a conductor, it will decay to zero exponentially with a decay time $\epsilon_b(0)/4\pi\sigma_0$. This is consistent with our assumption back in unit 2 that $\mathbf{E} = 0$ inside a conductor for electrostatics. Note, electrostatic fields obey $\mathbf{E} = -\boldsymbol{\nabla}\phi$ so for $\phi = \phi_\mathbf{k}\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}}$, then $\mathbf{E} = -i\mathbf{k}\,\phi_\mathbf{k}\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}}$ is longitudinal.

At high frequencies, $\omega \gg 1/\tau$ and $\omega \gg \omega_0$, we had,

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2} \qquad \text{so } \epsilon = 0 \text{ when} \quad \omega = \omega_p \tag{5.3.38}$$

So we have an oscillating longitudinal $\mathbf{E}$ only when $\omega = \omega_p$, independent of $\mathbf{k}$,

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_\omega \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}}\mathrm{e}^{i\omega_p t} \tag{5.3.39}$$

This is called a plasma oscillation. When one quantizes this oscillating mode it is called a *plasmon*. This oscillating longitudinal $\mathbf{E}$ is accompanied by an oscillating charge density,

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi\rho \quad \Rightarrow \quad \rho = \frac{i\mathbf{k}\cdot\mathbf{E}_\omega}{4\pi}\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}}\mathrm{e}^{-i\omega_p t} \tag{5.3.40}$$

So the plasma oscillation is a charge density oscillation (recall, for a transverse mode where $\mathbf{k}\cdot\mathbf{E}_\omega = 0$, the charge density vanishes).