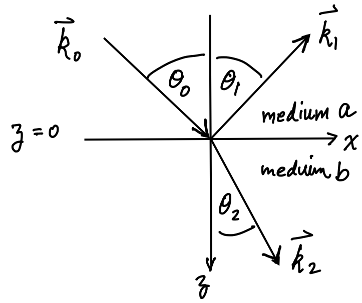
Unit 5-5-supp: Total Internal Reflection Revisited

Consider an EM wave propagating from medium a into medium b. We assume that ϵ_a and ϵ_b are real and positive, so that the media are transparent. We take μ_a and μ_b to be real and constant.

We will assume that $\epsilon_a > \epsilon_b$, or equivalently, that the indices of refraction $n_a = \sqrt{\mu_a \epsilon_a} > \sqrt{\mu_b \epsilon_b} = n_b$. There is thus a critical angle θ_c for total internal reflection, $\sin\theta_c = n_b/n_a$, for waves propagating from medium a into medium b. The geometry we consider is as in the diagram below. The plane of incidence is the xz plane.



$\mathbf{k}_0$ is the incident wave	θ_0 is the angle of incidence
$\mathbf{k}_1$ is the reflected wave	$\theta_1 = \theta_0$ is the angle of reflection
$\mathbf{k}_2$ is the transmitted wave	θ_2 is the angle of transmission

We know that $\omega_0 = \omega_1 = \omega_2 \equiv \omega$, and that $k_{0x} = k_{1x} = k_{2x}$ (the k_y components for each wave vanish).

From the dispersion relation in medium a we have,

$$k_0^2 = k_{0x}^2 + k_{0z}^2 = \frac{\omega^2}{c^2}\mu_a\epsilon_a = \frac{\omega^2}{c^2}n_a^2 \qquad \text{with } k_{0x} = k_0\sin\theta_0 \text{ and } k_{0z} = k_0\cos\theta_0. \tag{5.5.S.1}$$

From the dispersion relation in medium b we have (using $k_{2x} = k_{0x}$),

$$k_2^2 = k_{2x}^2 + k_{2z}^2 = k_{0x}^2 + k_{2z}^2 = \frac{\omega^2}{c^2}\mu_b\epsilon_b = \frac{\omega^2}{c^2}n_b^2 \tag{5.5.S.2}$$

So this gives,

$$k_{2z}^2 = \frac{\omega^2}{c^2}n_b^2 - k_{0x}^2 = \frac{\omega^2}{c^2}n_b^2 - k_0^2\sin^2\theta_0 = \frac{\omega^2}{c^2}n_b^2 - \frac{\omega^2}{c^2}n_a^2\sin^2\theta_0 = \frac{\omega^2}{c^2}\left[n_b^2 - n_a^2\sin^2\theta_0\right] \tag{5.5.S.3}$$

Now, writing $n_a\sin\theta_c = n_b$, we get,

$$k_{2z}^2 = \frac{\omega^2}{c^2}n_a^2\left[\sin^2\theta_c - \sin^2\theta_0\right] \tag{5.5.S.4}$$

1) When $\theta_0 < \theta_c$ we see that $k_{2z}^2 > 0$ and so k_{2z} is real valued, and the angle $\theta_2 = \arctan(k_{2x}/k_{2z})$ is well defined and in the region $(0, \pi/2)$. We therefore get Snell's Law,

$$k_{0x} = k_{2x} \quad \Rightarrow \quad k_0\sin\theta_0 = k_2\sin\theta_2 \quad \Rightarrow \quad n_a\sin\theta_0 = n_b\sin\theta_2 \tag{5.5.S.5}$$

2) However when $\theta_0 > \theta_c$, we see from Eq. (5.5.4) that $k_{2z}^2 < 0$ and so k_{2z} is imaginary,

$$k_{2z} = \frac{\omega}{c}n_a\sqrt{\left[\sin^2\theta_0 - \sin^2\theta_c\right]}\,i \tag{5.5.S.6}$$

This is total internal reflection. For $\theta_0 > \theta_c$ we see that the transmitted wave has a real valued component in the x-direction, $k_{2x} = k_{0x} = k_0\sin\theta_0$, but its component in the z-direction is pure imaginary, which means the amplitudes of the fields decay exponentially as z increases. The wave does not propagate into medium b.

We can define a penetration length by

$$\ell = \frac{1}{|k_{2z}|} = \frac{c}{\omega n_a}\frac{1}{\sqrt{\left[\sin^2\theta_0 - \sin^2\theta_c\right]}} \tag{5.5.S.7}$$

Just at $\theta_0 = \theta_c$ we have $\ell \to \infty$, reflecting that the wave does propagate into medium b as soon as θ_0 decreases below θ_c. As θ_0 increases above θ_c, ℓ decreases. Using $\sin\theta_c = n_b/n_a$, we find that the smallest value of ℓ, when $\theta_0 = \pi/2$, is

$$\ell_{\text{min}} = \frac{c}{\omega n_a} \frac{1}{\sqrt{1 - \sin^2\theta_c}} = \frac{c}{\omega n_a} \frac{1}{\cos\theta_c} = \frac{c}{\omega n_a} \frac{n_a}{\sqrt{n_a^2 - n_b^2}} = \frac{c}{\omega} \frac{1}{\sqrt{n_a^2 - n_b^2}} \tag{5.5.S.8}$$