Unit 6-4: Radiation in the Magnetic Dipole and Electric Quadrupole Approximations

In the long wavelength limit, the electric dipole radiation is the leading term and the higher terms are usually small and may be ignored. However it might happen that, for a particular charge configuration, the electric dipole moment vanishes, $\mathbf{p}_\omega = 0$. In this case it will be necessary to consider the magnetic dipole and electric quadrupole terms, which are both of the same order.

Magnetic Dipole Radiation

The magnetic dipole contribution to the vector potential, in the long wavelength approximation, is,

$$\mathbf{A}_{M1} = \frac{\mathrm{e}^{ikr}}{r}\left(\frac{1}{r} - ik\right)(-\hat{\mathbf{r}} \times \mathbf{m}_\omega) \tag{6.4.1}$$

In the Radiation Zone approximation, $kr \gg 1$, this becomes,

$$\mathbf{A}_{M1} = \frac{\mathrm{e}^{ikr}}{r}\, ik\,\hat{\mathbf{r}} \times \mathbf{m}_\omega \tag{6.4.2}$$

The magnetic field is,

$$\mathbf{B}_{M1} = \boldsymbol{\nabla} \times \mathbf{A}_{M1} = \left(\boldsymbol{\nabla}\,\mathrm{e}^{ikr}\right) \times \left(\frac{ik\hat{\mathbf{r}} \times \mathbf{m}_\omega}{r}\right) + \mathrm{e}^{ikr}\,\boldsymbol{\nabla} \times \left(\frac{ik\hat{\mathbf{r}} \times \mathbf{m}_\omega}{r}\right) \tag{6.4.3}$$

Taking the curl of the expression in the second term above gives terms of order $1/r^2$, and so these can be ignored in the Radiation Zone.

With $\boldsymbol{\nabla}\,\mathrm{e}^{ikr} = ik\hat{\mathbf{r}}\,\mathrm{e}^{ikr}$, we therefore get,

$$\boxed{\mathbf{B}_{M1} = -k^2\,\frac{\mathrm{e}^{ikr}}{r}\,\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{m}_\omega)} \qquad \text{in the Radiation Zone} \tag{6.4.4}$$

Far from the source, where $\mathbf{j} = 0$, Ampere's law gives,

$$\mathbf{E}_{M1} = \frac{i}{k}\,\boldsymbol{\nabla} \times \mathbf{B}_{M1} = -ik\left(\boldsymbol{\nabla}\,\mathrm{e}^{ikr}\right) \times \left(\frac{\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{m}_\omega)}{r}\right) - ik\,\mathrm{e}^{ikr}\,\boldsymbol{\nabla} \times \left(\frac{\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{m}_\omega)}{r}\right) \tag{6.4.5}$$

Taking the curl of the expression in the second term above gives terms of order $1/r^2$, and so these can be ignored in the Radiation Zone.

So using $\boldsymbol{\nabla}\,\mathrm{e}^{ikr} = ik\hat{\mathbf{r}}\,\mathrm{e}^{ikr}$, in the Radiation Zone we have,

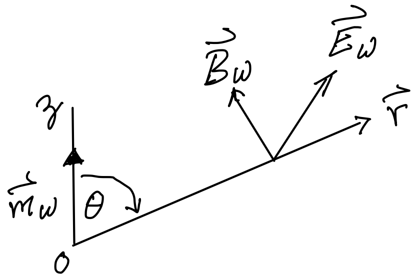
$$\mathbf{E}_{M1} = k^2\,\frac{\mathrm{e}^{ikr}}{r}\,\hat{\mathbf{r}} \times \Big(\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{m}_\omega)\Big) \tag{6.4.6}$$

Using $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$ with $\mathbf{C} = \hat{\mathbf{r}} \times \mathbf{m}_\omega$, we then get,

$$\mathbf{E}_{M1} = k^2\,\frac{\mathrm{e}^{ikr}}{r}\,\Big[\hat{\mathbf{r}}\Big(\hat{\mathbf{r}} \cdot (\hat{\mathbf{r}} \times \mathbf{m}_\omega)\Big) - (\hat{\mathbf{r}} \times \mathbf{m}_\omega)(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}})\Big] \tag{6.4.7}$$

$$\boxed{\mathbf{E}_{M1} = -k^2\,\frac{\mathrm{e}^{ikr}}{r}\,(\hat{\mathbf{r}} \times \mathbf{m}_\omega)} \qquad \text{in the Radiation Zone} \tag{6.4.8}$$

IF the dipole moment $\mathbf{m}_\omega$ is a real valued vector, then we can choose coordinates so that $\mathbf{m}_\omega = m_\omega\hat{\mathbf{z}}$ is aligned along the $\hat{\mathbf{z}}$ axis. Then we can write in spherical coordinates,



$$\mathbf{E}_{M1}(\mathbf{r}) = k^2\, m_\omega \,\frac{e^{ikr}}{r}\,\sin\theta\,\hat{\boldsymbol{\varphi}}$$
$$\mathbf{B}_{M1}(\mathbf{r}) = -k^2\, m_\omega \,\frac{e^{ikr}}{r}\,\sin\theta\,\hat{\boldsymbol{\theta}} \tag{6.4.9}$$

The Poynting vector is,

$$\mathbf{S}_{M1}(\mathbf{r},t) = \frac{c}{4\pi}\mathrm{Re}\left[\mathbf{E}_{M1}(\mathbf{r})e^{-i\omega t}\right]\times \mathrm{Re}\left[\mathbf{B}_{M1}(\mathbf{r})e^{-i\omega t}\right] \tag{6.4.10}$$

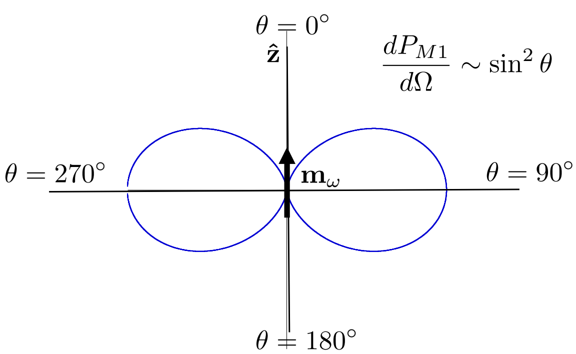
$$= \frac{c}{4\pi}\,\frac{k^4\, m_\omega^2}{r^2}\,\cos^2(kr-\omega t)\,\sin^2\theta\,\hat{\mathbf{r}} \tag{6.4.11}$$

And the time averaged Poynting vector is,

$$\langle \mathbf{S}_{M1}\rangle = \frac{c}{8\pi}\,\frac{k^4\, m_\omega^2}{r^2}\sin^2\theta\,\hat{\mathbf{r}} \tag{6.4.12}$$

This is exactly the same form as we found previously for the electric dipole $\langle \mathbf{S}_{E1}\rangle$ except with the replacement $\mathbf{p}_\omega \to \mathbf{m}_\omega$. Therefore we have,

$$\frac{dP_{M1}}{d\Omega} = \hat{\mathbf{r}}\cdot\langle \mathbf{S}_{M1}\rangle\, r^2 = \frac{c}{8\pi}\,k^4\, m_\omega^2\,\sin^2\theta \sim \omega^4\,\sin^2\theta \tag{6.4.13}$$



The power distribution is as shown on the left. The distribution is rotationally symmetric about the $\hat{\mathbf{z}}$ axis. Most of the power is directed outwards into the xy plan $\perp\, \mathbf{m}_\omega$.

The total radiated power is,

$$P_{M1} = \int d\Omega\,\frac{dP_{M1}}{d\Omega} = \frac{c\,k^4\, m_\omega^2}{3} = \frac{m_\omega^2\omega^4}{3\,c^3} \sim \omega^4 \tag{6.4.14}$$

The ratio of power emitted in magnetic dipole vs electric dipole radiation is,

$$\frac{P_{M1}}{P_{E1}} = \frac{m_\omega^2}{p_\omega^2} \sim \left(\frac{v}{c}\right)^2 \qquad \text{since } m_\omega \sim \frac{d\,j}{c} \sim dq\frac{v}{c} \text{ and } p_\omega \sim dq \tag{6.4.15}$$

Electric Quadrupole Radiation

The electric quadrupole contribution to the vector potential, in the long wavelength approximation, is.

$$\mathbf{A}_{E2} = \frac{\mathrm{e}^{ikr}}{r}\left(\frac{1}{r} - ik\right)\left(-\frac{i\omega}{6c}\hat{\mathbf{r}}\cdot\overleftrightarrow{\mathbf{Q}}_\omega\right) \tag{6.4.16}$$

In the Radiation Zone approximation, $kr \gg 1$, this becomes,

$$\mathbf{A}_{E2} = -\frac{\mathrm{e}^{ikr}}{r}\frac{k^2}{6}\hat{\mathbf{r}}\cdot\overleftrightarrow{\mathbf{Q}}_\omega \qquad \text{using } k = \omega/c \tag{6.4.17}$$

The magnetic field is,

$$\mathbf{B}_{E2} = \boldsymbol{\nabla}\times\mathbf{A}_{E2} = -\left(\boldsymbol{\nabla}\,\mathrm{e}^{ikr}\right)\times\left(\frac{k^2\,\hat{\mathbf{r}}\cdot\overleftrightarrow{\mathbf{Q}}_\omega}{6r}\right) - \mathrm{e}^{ikr}\,\boldsymbol{\nabla}\times\left(\frac{k^2\,\hat{\mathbf{r}}\cdot\overleftrightarrow{\mathbf{Q}}_\omega}{6r}\right) \tag{6.4.18}$$

Taking the curl of the expression in the second term above gives terms of order $1/r^2$, and so these can be ignored in the Radiation Zone.

With $\boldsymbol{\nabla}\,\mathrm{e}^{ikr} = ik\hat{\mathbf{r}}\,\mathrm{e}^{ikr}$, we therefore get,

$$\boxed{\mathbf{B}_{E2} = -ik^3\,\frac{\mathrm{e}^{ikr}}{6r}\,\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\cdot\overleftrightarrow{\mathbf{Q}}_\omega\right)} \qquad \text{in the Radiation Zone} \tag{6.4.19}$$

Far from the source, where $\mathbf{j} = 0$, Ampere's law gives,

$$\mathbf{E}_{E2} = \frac{i}{k}\,\boldsymbol{\nabla}\times\mathbf{B}_{E2} = k^2\left(\boldsymbol{\nabla}\,\mathrm{e}^{ikr}\right)\times\left(\frac{\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\cdot\overleftrightarrow{\mathbf{Q}}_\omega)}{6r}\right) + k^2\,\mathrm{e}^{ikr}\,\boldsymbol{\nabla}\times\left(\frac{\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\cdot\overleftrightarrow{\mathbf{Q}}_\omega)}{6r}\right) \tag{6.4.20}$$

Taking the curl of the expression in the second term above gives terms of order $1/r^2$, and so these can be ignored in the Radiation Zone.

So using $\boldsymbol{\nabla}\,\mathrm{e}^{ikr} = ik\hat{\mathbf{r}}\,\mathrm{e}^{ikr}$, in the Radiation Zone we have,

$$\boxed{\mathbf{E}_{E2} = ik^3\,\frac{e^{ikr}}{6r}\,\hat{\mathbf{r}}\times\left(\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\cdot\overleftrightarrow{\mathbf{Q}}_\omega)\right)} \qquad \text{in the Radiation Zone} \tag{6.4.21}$$

The Poynting vector is,

$$\mathbf{S}_{E2}(\mathbf{r},t) = \frac{c}{4\pi}\mathrm{Re}\,\left[\mathbf{E}_{E2}(\mathbf{r})\mathrm{e}^{-i\omega t}\right]\times\mathrm{Re}\,\left[\mathbf{B}_{E2}(\mathbf{r})\mathrm{e}^{-i\omega t}\right] \tag{6.4.22}$$

$$= -\frac{c}{4\pi}\frac{k^6}{36r^2}\sin^2(kr-\omega t)\left[\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\cdot\overleftrightarrow{\mathbf{Q}}_\omega))\right]\times\left[\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\cdot\overleftrightarrow{\mathbf{Q}}_\omega)\right] \tag{6.4.23}$$

where we have assumed that $\overleftrightarrow{\mathbf{Q}}_\omega$ is real.

Let $\mathbf{W} \equiv \hat{\mathbf{r}}\times(\hat{\mathbf{r}}\cdot\overleftrightarrow{\mathbf{Q}}_\omega)$. Then,

$$\left[\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\cdot\overleftrightarrow{\mathbf{Q}}_\omega))\right]\times\left[\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\cdot\overleftrightarrow{\mathbf{Q}}_\omega)\right] = [\hat{\mathbf{r}}\times\mathbf{W}]\times\mathbf{W} = \mathbf{W}\times(\mathbf{W}\times\hat{\mathbf{r}}) = \mathbf{W}(\mathbf{W}\cdot\hat{\mathbf{r}}) - \hat{\mathbf{r}}\,(\mathbf{W}\cdot\mathbf{W}) \tag{6.4.24}$$

Now,

$$\mathbf{W}\cdot\hat{\mathbf{r}} = \left[\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\cdot\overleftrightarrow{\mathbf{Q}}_\omega)\right]\cdot\hat{\mathbf{r}} = 0 \qquad \text{since } \hat{\mathbf{r}}\perp(\hat{\mathbf{r}}\times\mathbf{U}) \text{ for any } \mathbf{U} \tag{6.4.25}$$

Now, letting $\mathbf{U} \equiv \hat{\mathbf{r}} \cdot \overleftrightarrow{\mathbf{Q}}_\omega$ so that $\mathbf{W} = \hat{\mathbf{r}} \times \mathbf{U}$, and using $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})$ and $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$, we have,

$$\left[\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \cdot \overleftrightarrow{\mathbf{Q}}_\omega))\right] \times \left[\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \cdot \overleftrightarrow{\mathbf{Q}}_\omega)\right] = -\hat{\mathbf{r}}\,(\mathbf{W} \cdot \mathbf{W}) = -\hat{\mathbf{r}}\,(\hat{\mathbf{r}} \times \mathbf{U}) \cdot (\hat{\mathbf{r}} \times \mathbf{U}) = -\hat{\mathbf{r}}\,\hat{\mathbf{r}} \cdot [\mathbf{U} \times (\hat{\mathbf{r}} \times \mathbf{U})] \tag{6.4.26}$$

$$= -\hat{\mathbf{r}}\,\hat{\mathbf{r}} \cdot [\hat{\mathbf{r}}\,(\mathbf{U} \cdot \mathbf{U}) - \mathbf{U}\,(\mathbf{U} \cdot \hat{\mathbf{r}})] = -\hat{\mathbf{r}}\,[(\mathbf{U} \cdot \mathbf{U}) - (\hat{\mathbf{r}} \cdot \mathbf{U}) \cdot (\mathbf{U} \cdot \hat{\mathbf{r}})] = -\left[|\mathbf{U}|^2 - |\hat{\mathbf{r}} \cdot \mathbf{U}|^2\right]\hat{\mathbf{r}} \tag{6.4.27}$$

$$= -\left[|\hat{\mathbf{r}} \cdot \overleftrightarrow{\mathbf{Q}}_\omega|^2 - |\hat{\mathbf{r}} \cdot \overleftrightarrow{\mathbf{Q}}_\omega \cdot \hat{\mathbf{r}}|^2\right]\hat{\mathbf{r}} \tag{6.4.28}$$

So finally,

$$\mathbf{S}_{E2} = \frac{c}{4\pi}\frac{k^6}{36r^2}\sin^2(kr - \omega t)\left[|\hat{\mathbf{r}} \cdot \overleftrightarrow{\mathbf{Q}}_\omega|^2 - |\hat{\mathbf{r}} \cdot \overleftrightarrow{\mathbf{Q}}_\omega \cdot \hat{\mathbf{r}}|^2\right]\hat{\mathbf{r}} \tag{6.4.29}$$

Taking the time average gives,

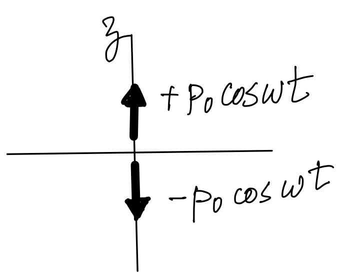
$$\langle \mathbf{S}_{E2} \rangle = \frac{c}{4\pi}\frac{k^6}{72r^2}\left[|\hat{\mathbf{r}} \cdot \overleftrightarrow{\mathbf{Q}}_\omega|^2 - |\hat{\mathbf{r}} \cdot \overleftrightarrow{\mathbf{Q}}_\omega \cdot \hat{\mathbf{r}}|^2\right]\hat{\mathbf{r}} \tag{6.4.30}$$

and then the differential power cross-section is,

$$\frac{dP_{E2}}{d\Omega} = \hat{\mathbf{r}} \cdot \langle \mathbf{S}_{E2} \rangle\, r^2 = \frac{c}{4\pi}\frac{k^6}{72}\left[|\hat{\mathbf{r}} \cdot \overleftrightarrow{\mathbf{Q}}_\omega|^2 - |\hat{\mathbf{r}} \cdot \overleftrightarrow{\mathbf{Q}}_\omega \cdot \hat{\mathbf{r}}|^2\right] \tag{6.4.31}$$

The angular dependence of $dP_{E2}/d\Omega$ depends on the specific form of $\overleftrightarrow{\mathbf{Q}}_\omega$.

As an example, suppose $\hat{\mathbf{e}}_i \cdot \overleftrightarrow{\mathbf{Q}}_\omega \cdot \hat{\mathbf{e}}_j = 0$ except for $\hat{\mathbf{e}}_3 \cdot \overleftrightarrow{\mathbf{Q}}_\omega \cdot \hat{\mathbf{e}}_3 \equiv Q_{zz}$, so that $\overleftrightarrow{\mathbf{Q}}_\omega = Q_{zz}\,\hat{\mathbf{z}}\,\hat{\mathbf{z}}$. This would be a model for two equal, but oppositely oriented, oscillating electric dipoles.



Then,

$$\hat{\mathbf{r}} \cdot \overleftrightarrow{\mathbf{Q}}_\omega = Q_{zz}\,(\hat{\mathbf{r}} \cdot \hat{\mathbf{z}})\,\hat{\mathbf{z}} = Q_{zz}\cos\theta\,\hat{\mathbf{z}} \tag{6.4.32}$$

$$|\hat{\mathbf{r}} \cdot \overleftrightarrow{\mathbf{Q}}_\omega|^2 = Q_{zz}^2\cos^2\theta \tag{6.4.33}$$

$$\hat{\mathbf{r}} \cdot \overleftrightarrow{\mathbf{Q}}_\omega \cdot \hat{\mathbf{r}} = Q_{zz}(\hat{\mathbf{r}} \cdot \hat{\mathbf{z}})(\hat{\mathbf{z}} \cdot \hat{\mathbf{r}}) = Q_{zz}\cos^2\theta \tag{6.4.34}$$

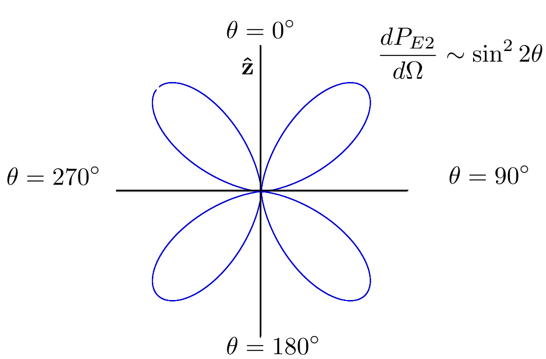
$$|\hat{\mathbf{r}} \cdot \overleftrightarrow{\mathbf{Q}}_\omega \cdot \hat{\mathbf{r}}|^2 = Q_{zz}^2\cos^4\theta \tag{6.4.35}$$

and so,

$$\frac{dP_{E2}}{d\Omega} = \frac{c}{4\pi}\frac{k^6}{72}Q_{zz}^2\left[\cos^2\theta - \cos^4\theta\right] = \frac{c}{4\pi}\frac{k^6}{72}Q_{zz}^2\,\cos^2\theta\,\left[1 - \cos^2\theta\right] = \frac{c}{4\pi}\frac{k^6}{72}Q_{zz}^2\,\cos^2\theta\,\sin^2\theta \tag{6.4.36}$$

and, using $\sin 2\theta = 2\,\sin\theta\cos\theta$, we finally have,

$$\frac{dP_{E2}}{d\Omega} = \frac{c}{4\pi}\frac{k^6}{288}Q_{zz}^2\,\sin^2 2\theta \tag{6.4.37}$$



The power distribution is as shown on the left. The distribution is rotationally symmetric about the $\hat{\mathbf{z}}$ axis. Most of the powser is directed at $\pm 45°$ to the $\hat{\mathbf{z}}$ axis.

The total radiated power is,

$$P_{E2} = \int d\Omega\, \frac{dP_{E2}}{d\Omega} \sim k^6 Q^2 \sim k^6\,(qd^2)^2 \tag{6.4.38}$$

So the ratio of electric quadrupole power to electric dipole power is,

$$\frac{P_{E2}}{P_{E1}} \sim \frac{k^6 Q^2}{k^4 p^2} \sim \frac{k^6 (qd^2)^2}{k^4 (qd)^2} \sim (kd)^2 \sim \left(\frac{v}{c}\right)^2 \tag{6.4.39}$$

since $kd = (\omega/c)d \sim (d/\tau)/c \sim (v/c)$. So $P_{E2} \sim P_{M1}$.

General Quardupole

For the most general electric quadrupole case, since $\overleftrightarrow{\mathbf{Q}}_\omega$ is symmetric, we can choose coordinate axes so that $\overleftrightarrow{\mathbf{Q}}_\omega$ is diagonal. Then, with $\hat{\mathbf{r}} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, we have,

$$\hat{\mathbf{r}} \cdot \overleftrightarrow{\mathbf{Q}}_\omega \cdot \hat{\mathbf{r}} \;=\; \hat{\mathbf{r}} \cdot \begin{pmatrix} Q_{xx} & 0 & \\ 0 & Q_{yy} & 0 \\ 0 & 0 & Q_{zz} \end{pmatrix} \cdot \hat{\mathbf{r}} \;=\; Q_{xx}\sin^2\theta\cos^2\varphi + Q_{yy}\sin^2\theta\sin^2\varphi + Q_{zz}\cos^2\theta \tag{6.4.40}$$

and

$$|\hat{\mathbf{r}} \cdot \overleftrightarrow{\mathbf{Q}}_\omega|^2 \;=\; Q_{xx}^2\sin^2\theta\cos^2\varphi + Q_{yy}^2\sin^2\theta\sin^2\varphi + Q_{zz}^2\cos^2\theta \tag{6.4.41}$$

so

$$\frac{dP_{E2}}{d\Omega} = \frac{c}{4\pi}\frac{k^6}{72}\Big[Q_{xx}^2\sin^2\theta\cos^2\varphi + Q_{yy}^2\sin^2\theta\sin^2\varphi + Q_{zz}^2\cos^2\theta \tag{6.4.42}$$

$$- \left(Q_{xx}\sin^2\theta\cos^2\varphi + Q_{yy}\sin^2\theta\sin^2\varphi + Q_{zz}\cos^2\theta\right)^2\Big] \tag{6.4.43}$$

In general, if there are no special symmetries, $dP_{E2}/d\Omega$ varies with both θ and φ.

Discussion Question 6.4

Consider the radiation emitted by a thin circular wire loop of radius R, centered about the origin in the xy plane at $z = 0$. The current flowing in the loop is given by

$$I(\varphi, t) = \mathrm{Re}\left[I_0\cos(n\varphi)\mathrm{e}^{-i\omega t}\right]$$

where φ is the usual azimuthal angle in spherical coordinates (i.e. the angle in the xy plane). The frequency ω is such that $R\omega \ll c$.

What type of radiation (i.e. electric dipole, magnetic dipole, electric quadrupole, etc.) is emitted when $n = 0$, $n = 1$, and $n = 2$.

You should *not* need to do any analytic calculation, just think of how the charge in the wire loop is behaving when the current is as specified. A well explained sketch should be sufficient to make your point.