Unit 7-2: Maxwell's Equations in Relativistic Form

The theory of electromagnetism is intimately tied to the theory of special relativity, and indeed formed the motivation for Einstein to develop his theory. From our study of electromagnetism, we can easily see several hints of this.

Consider a charged particle q moving with a constant velocity $\mathbf{v}$. The charge feels a Lorentz force,

$$\mathbf{F} = q\mathbf{E} + q\frac{\mathbf{v}}{c} \times \mathbf{B} \tag{7.2.1}$$

The magnetic part of the force is determined by the charge's velocity $\mathbf{v}$. But what is that velocity measured with respect to? According to special relativity, there is no absolute "rest frame." Or alternatively, there is no way to determine a unique value of the charge's velocity – as we look at the moving charge in different inertial frames of reference, its velocity will change, and the force on it from the magnetic field will change. Indeed, if we move to the inertial frame of reference in which the charge is at rest, then there is no magnetic force on it at all!

Similarly, the values of the electromagnetic fields themselves depend on which inertial frame of reference we are in. Consider an infinite straight line with fixed charge per unit length λ, which is at rest in inertial frame $\mathcal{K}$. In the frame $\mathcal{K}$ there is an electric field $\mathbf{E}$ pointing outward in the cylindrical radial direction from the line charge. There is no magnetic field $\mathbf{B} = 0$.

Now make a transformation to an inertial frame of reference $\mathcal{K}'$ that is moving with velocity $\mathbf{v}$ in a direction parallel to the line charge, as seen by $\mathcal{K}$. In this frame, the line charge is moving with velocity $-\mathbf{v}$, and so carries a current $\mathbf{I} = -\lambda\mathbf{v}$. This current gives rise to a magnetic field $\mathbf{B}'$ that circulates around the line charge, as well as a radially outward pointing $\mathbf{E}'$.

Clearly $\mathbf{E}$ and $\mathbf{B}$ must change from one inertial frame of reference to another. In this section we therefore see how to recast electromagnetism in a form that is manifestly compatible with special relativity. In doing so, we will see that $\mathbf{E}$ and $\mathbf{B}$ should not be viewed as separate quantities, but rather are both parts of a new relativistic field tensor.

4-Current

Consider the total charge ΔQ contained in a small box of volume ΔV. ΔQ is a Lorentz invariant scalar. In another inertial reference frame, even though the shape of the box may deform, the total charge within the box must stay the same.

Consider the frame of reference in which this charge ΔQ is at rest. We will denote quantities measured in this frame by a circle over the quantity. Let $\Delta\mathring{V}$ be the volume of the box in this rest frame, and $\mathring{\rho}$ the charge density. Then,

$$\Delta Q = \mathring{\rho}\,\Delta\mathring{V} \tag{7.2.2}$$

By definition, $\mathring{\rho}$ is a Lorentz invariant scalar, because it is defined as the density in the charge's rest frame. Now transform to another inertial frame of reference moving with velocity $\mathbf{v}$ with respect to the rest frame. In this new frame, ΔQ stays the same since it is a Lorentz invariant scalar. The volume becomes,

$$\Delta V = \frac{\Delta\mathring{V}}{\gamma} \tag{7.2.3}$$

since the length of the box in the direction parallel to $\mathbf{v}$ contracts by a factor $1/\gamma$, while the transverse directions stay the same. Thus the charge density in this new frame of reference is,

$$\rho = \frac{\Delta Q}{\Delta V} = \frac{\Delta Q}{\Delta\mathring{V}}\gamma = \mathring{\rho}\,\gamma \tag{7.2.4}$$

The current density in this new frame is $\mathbf{j} = \rho\mathbf{v} = (\rho/\gamma)(\gamma\mathbf{v}) = \mathring{\rho}\,\mathbf{u}$, where $\mathbf{u}$ is the spatial parts of the 4-velocity.

We therefore define the 4-current as,

$$\boxed{j_\mu \equiv (\mathbf{j}, ic\rho) = \mathring{\rho}\,(\mathbf{u}, ic\gamma) = \mathring{\rho}\,u_\mu} \qquad \text{using } \rho = \mathring{\rho}\,\gamma \text{ and } \mathbf{j} = \mathring{\rho}\,\mathbf{u} \tag{7.2.5}$$

Since $\mathring{\rho}$ is a Lorentz invariant scalar, and u_μ is a 4-vector, so j_μ is a 4-vector.

We can now write the law of charge conservation as,

$$\boldsymbol{\nabla}\cdot\mathbf{j}+\frac{\partial\rho}{\partial t} = \boxed{\frac{\partial j_\mu}{\partial x_\mu} = 0} \tag{7.2.6}$$

4-Potential

The equations for the electromagnetic potentials in the Lorenz gauge are,

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A}=-\frac{4\pi}{c}\mathbf{j} \tag{7.2.7}$$

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi=-4\pi\rho=-\frac{4\pi}{c}(-i)j_4 \qquad\Rightarrow\qquad \left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)i\phi=-\frac{4\pi}{c}j_4 \tag{7.2.8}$$

As we demonstrated above, the wave equation operator is the Lorentz invariant scalar differential operator $\partial^2/\partial x_\nu^2$.

We now define the 4-potential

$$\boxed{A_\mu\equiv(\mathbf{A},i\phi)} \tag{7.2.9}$$

Eqs. (7.2.7) and (7.2.8) then become,

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)A_\mu = \boxed{\frac{\partial^2 A_\mu}{\partial x_\nu^2} = -\frac{4\pi}{c}j_\mu} \tag{7.2.10}$$

Since $\partial^2/\partial x_\nu^2$ is a Lorentz invariant scalar, and j_μ is a 4-vector, it follows that A_μ must be a 4-vector.

The Lorenz gauge condition now becomes,

$$\boldsymbol{\nabla}\cdot\mathbf{A}+\frac{1}{c}\frac{\partial\phi}{\partial t} = \boxed{\frac{\partial A_\mu}{\partial x_\mu} = 0} \tag{7.2.11}$$

Since A_μ and $\partial/\partial x_\mu$ are both 4-vectors, their inner product is a Lorentz invariant scalar. We thus see that if the potentials obey the Lorenz gauge condition in one inertial frame of reference, then they will obey the Lorenz gauge condition in all inertial frames of reference, since $\partial A'_\mu/\partial x'_\mu=\partial A_\mu/\partial x_\mu$ is a Lorentz invariant scalar.

We have thus completely recast Maxwell's equations, written in terms of the vector and scalar potentials $\mathbf{A}$ and ϕ, into a relativistic formulation written in terms of the 4-vectors A_μ and j_μ. We now want to see how to formulate the electric and magnetic fields in a relativistic way.

The Field Strength Tensor

The magnetic field is obtained from the vector potential by,

$$B_i=\frac{\partial A_k}{\partial x_j}-\frac{\partial A_j}{\partial x_k} \qquad \text{with } i,j,k \text{ a cyclic permutation of 1,2,3} \tag{7.2.12}$$

The electric field is obtained from the scalar and vector potentials by,

$$E_i=-\frac{\partial\phi}{\partial x_i}-\frac{\partial A_i}{c\,\partial t}=i\left(\frac{\partial A_4}{\partial x_i}-\frac{\partial A_i}{\partial x_4}\right) \tag{7.2.13}$$

We therefore define the field strength tensor,

$$\boxed{F_{\mu\nu}\equiv\frac{\partial A_\nu}{\partial x_\mu}-\frac{\partial A_\mu}{\partial x_\nu}=-F_{\nu\mu}} \tag{7.2.14}$$

$F_{\mu\nu}$ is the 4 dimensional analog of the curl of a 3-vector. It is an antisymmetric 2nd rank tensor. In general the analog of curl (or cross product), in any dimension higher than 3, is such an antisymmetric 2nd rank tensor.

In 3 dimensions, the analogous antisymmetric 2nd rank tensor $(\partial A_k/\partial x_j)-(\partial A_j/\partial x_k)$ has 3 independent components, which can thus be associated with the three components of the magnetic field vector $\mathbf{B}$ (actually a psuedovector). But in 4 dimensions, the antisymmetric 2nd rank tensor $F_{\mu\nu} = (\partial A_\nu/\partial x_\mu) - (\partial A_\mu/\partial x_\nu)$ has 6 independent components, and so cannot be associated with any 4-vector (or 4-psuedovector). However, the 6 independent components of $F_{\mu\nu}$ are just the right number of components to represent the electric and magnetic field 3-vectors!

From Eqs. (7.2.14) and (7.2.9), or equivalently from Eqs. (7.2.12) and (7.2.13) we get,

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{pmatrix} \tag{7.2.15}$$

$F_{\mu\nu}$ is a 2nd rank 4-tensor. To see how it transforms under a Lorentz transformation $a(\mathcal{L})$, we note,

$$F'_{\mu\nu} = \frac{\partial A'_\nu}{\partial x'_\mu} - \frac{\partial A'_\mu}{\partial x'_\nu} \qquad \text{Use} \quad A'_\mu = a_{\mu\sigma}A_\sigma \quad \text{and} \quad \frac{\partial}{\partial x'_\mu} = a_{\mu\lambda}\frac{\partial}{\partial x_\lambda} \quad \text{since both are 4-vectors} \tag{7.2.16}$$

and so

$$F'_{\mu\nu} = a_{\nu\lambda}a_{\mu\sigma}\frac{\partial A_\lambda}{\partial x_\sigma} - a_{\mu\sigma}a_{\nu\lambda}\frac{\partial A_\sigma}{\partial x_\lambda} = a_{\mu\sigma}a_{\nu\lambda}\left(\frac{\partial A_\lambda}{\partial x_\sigma} - \frac{\partial A_\sigma}{\partial x_\lambda}\right) = a_{\mu\sigma}a_{\nu\lambda}F_{\sigma\lambda} \tag{7.2.17}$$

The Lorentz transformation law for an nth rank 4-tensor is similarly defined as,

$$T'_{\mu_1\mu_2\mu_2\ldots\mu_n} = a_{\mu_1\nu_1}a_{\mu_2\nu_2}a_{\mu_3\nu_3}\ldots a_{\mu_n\nu_n}T_{\nu_1\nu_2\nu_2\ldots\nu_n} \tag{7.2.18}$$

From Eq. (7.2.17) we can get how the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ transform when going from one inertial frame of reference to another. We will come back to that at the end of this section.

Maxwell's Equations

Finally we want to write Maxwell's equations, using our new field strength tensor $F_{\mu\nu}$ to represent the fields $\mathbf{E}$ and $\mathbf{B}$. Note, Maxwell's equations are 1st order linear partial differential equations for the fields.

Maxwell's inhomogeneous equations

It is easy to find the proper equation for the *inhomogeneous* Maxwell's equations. These are,

$$\boldsymbol{\nabla}\times\mathbf{B} - \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{j} \qquad \text{and} \qquad \boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi\rho \tag{7.2.19}$$

The first, Ampere's law, is a 3-vector equation and so represents 3 scalar equations. The second, Gauss' law, is a scalar equation. So altogether there are 4 inhomogeneous equations, the same number of components as a 4-vector! What 1st order linear differential equation for a 4-vector can we construct from the 2nd rank 4-tensor $F_{\mu\nu}$? The natural thing to guess is the inner product of the 4-gradient $\partial/\partial x_\nu$ with $F_{\mu\nu}$. Using the definition of $F_{\mu\nu}$ as the "curl" of A_μ, as in Eq. (7.2.14), we have,

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \frac{\partial}{\partial x_\nu}\left(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}\right) = \frac{\partial^2 A_\nu}{\partial x_\nu\partial x_\mu} - \frac{\partial^2 A_\mu}{\partial x_\nu^2} = \frac{\partial}{\partial x_\mu}\left(\frac{\partial A_\nu}{\partial x_\nu}\right) - \frac{\partial^2 A_\mu}{\partial x_\nu^2} = 0 - \frac{\partial^2 A_\mu}{\partial x_\nu^2} \tag{7.2.20}$$

where we used the fact that A_ν obeys the Lorenz gauge condition and so $\partial A_\nu/\partial x_\nu = 0$. But from Eq. (7.2.10) we have, $\partial^2 A_\mu/\partial x_\nu^2 = -(4\pi/c)j_\mu$. We therefore conclude that the inhomogeneous Maxwell's equations can be written in terms of the field strength tensor as,

$$\boxed{\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \frac{4\pi}{c} j_\mu} \qquad \text{gives Maxwell's inhomogeneous equations} \tag{7.2.21}$$

Note, we can directly show that $\partial F_{\mu\nu}/\partial x_\nu$ is a 4-vector, by looking how it transforms under a Lorentz transformation.

$$\frac{\partial F'_{\mu\nu}}{\partial x'_\nu} = a_{\mu\sigma} a_{\nu\lambda} a_{\nu\tau} \frac{\partial F_{\sigma\lambda}}{\partial x_\tau} \tag{7.2.22}$$

which follows since $F_{\mu\nu}$ transforms as a 2nd rank tensor, and $\dfrac{\partial}{\partial x_\nu}$ transforms as a 4-vector.

But $a_{\nu\lambda} = a^{-1}_{\lambda\nu}$ since the transpose of the matrix is its inverse. Then $a_{\nu\lambda} a_{\nu\tau} = a^{-1}_{\lambda\nu} a_{\nu\tau} = \delta_{\lambda\tau}$. So we have,

$$\frac{\partial F'_{\mu\nu}}{\partial x'_\nu} = a_{\mu\sigma} a_{\nu\lambda} a_{\nu\tau} \frac{\partial F_{\sigma\lambda}}{\partial x_\tau} = a_{\mu\sigma} \delta_{\lambda\tau} \frac{\partial F_{\sigma\lambda}}{\partial x_\tau} = a_{\mu\sigma} \frac{\partial F_{\sigma\lambda}}{\partial x_\lambda} \tag{7.2.23}$$

which is just the transformation law for a 4-vector.

Maxwell's homogeneous equations

We now wish to express the *homogeneous* Maxwell's equations in terms of $F_{\mu\nu}$. This is a little trickier, and we will show two different, but equivalent, ways to do it. The homogeneous Maxwell's equations are,

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \text{and} \qquad \boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{7.2.24}$$

These consist of one 3-vector equation and one scalar equation, representing a total of 4 scalar equations.

From the field strength tensor $F_{\mu\nu}$ we can construct a 3rd rank 4-tensor,

$$\boxed{G_{\mu\nu\lambda} = \frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x_\nu} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu}} \tag{7.2.25}$$

In each term, the indicies are a cyclic permutation of the indicies in the preceding term.

This 3rd rank tensor transforms as,

$$G'_{\mu\nu\lambda} = a_{\mu\sigma} a_{\nu\tau} a_{\lambda\eta} G_{\sigma\tau\eta} \tag{7.2.26}$$

In principle, $G_{\mu\nu\lambda}$ has $4^3 = 64$ components. However we can show that $G_{\mu\nu\lambda}$ is antisymmetric in the exhange of any two of its indices. We show this by making use of the fact that $F_{\mu\nu} = -F_{\nu\mu}$ is antisymmetric,

$$G_{\mu\nu\lambda} = \frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x_\nu} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu} = -\frac{\partial F_{\nu\mu}}{\partial x_\lambda} - \frac{\partial F_{\mu\lambda}}{\partial x_\nu} - \frac{\partial F_{\lambda\nu}}{\partial x_\mu} = -G_{\nu\mu\lambda} = -G_{\mu\lambda\nu} = -G_{\lambda\nu\mu} \tag{7.2.27}$$

Since $G_{\mu\nu\lambda}$ is antisymmetric in the exchange of any of its two indices, it is only non-zero when all three indices are different. Since each index can take only the values 1,2,3 or 4, there are only 4 independent components,

$$G_{412}, \quad G_{413}, \quad G_{423}, \quad G_{123} \tag{7.2.28}$$

All the other components either vanish or are $\pm$ one of the above. These 4 independent components are just the right number to give the 4 homogeneous Maxwell's equations! They can be written as,

$$\boxed{G_{\mu\nu\lambda} = 0} \qquad \text{gives Maxwell's homogeneous equations} \tag{7.2.29}$$

To see that $G_{\mu\nu\lambda}$ must indeed vanish, we can substitute into $G_{\mu\nu\lambda}$ the definition of $F_{\mu\nu}$ in terms of the A_μ. We get,

$$G_{\mu\nu\lambda} = \frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x_\nu} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu} = \frac{\partial^2 A_\nu}{\partial x_\lambda \partial x_\mu} - \frac{\partial^2 A_\mu}{\partial x_\lambda \partial x_\nu} + \frac{\partial^2 A_\mu}{\partial x_\nu \partial x_\lambda} - \frac{\partial^2 A_\lambda}{\partial x_\nu \partial x_\mu} + \frac{\partial^2 A_\lambda}{\partial x_\mu \partial x_\nu} - \frac{\partial^2 A_\nu}{\partial x_\mu \partial x_\lambda} \tag{7.2.30}$$

Since the order in which we take the derivatives does not matter, i.e. $\partial/\partial x_\lambda \partial x_\mu = \partial/\partial x_\mu \partial x_\lambda$ we see that all the terms cancel in pairs, and so $G_{\mu\nu\lambda} = 0$.

One can show that,

$$G_{123} = \boldsymbol{\nabla}\cdot\mathbf{B} = 0 \qquad \text{and} \qquad G_{412} = -i\left[\boldsymbol{\nabla}\times\mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}\right]_z = 0 \tag{7.2.31}$$

Similarly, $G_{413} = 0$ and $G_{423} = 0$ give the y and x components of Faraday's law.

There is another, more common, way to write the homogeneous Maxwell's equations. This is obtained by noting that we can get the homogeneous equations from the inhomogeneous equations by taking the sources $\rho = \mathbf{j} = 0$ and making the substitutions $\mathbf{B} \to -\mathbf{E}$ and $\mathbf{E} \to \mathbf{B}$. Making these substitutions within the field strength tenser $F_{\mu\nu}$ then defines the *dual field strength tensor*

$$\tilde{F}_{\mu\nu} = \begin{pmatrix} 0 & -E_3 & E_2 & -iB_1 \\ E_3 & 0 & -E_1 & -iB_1 \\ -E_2 & E_1 & 0 & -iB_3 \\ iB_1 & iB_2 & iB_3 & 0 \end{pmatrix} \tag{7.2.32}$$

in terms of which the homogeneous Maxwell's equations can then be written as,

$$\boxed{\frac{\partial\tilde{F}_{\mu\nu}}{\partial x_\nu} = 0} \qquad \text{gives Maxwell's homogeneous equations} \tag{7.2.33}$$

To make our definition of $\tilde{F}_{\mu\nu}$ a bit more formal (so as to see that it is an appropriate 4-tensor), we define the 4 dimensional analog of the Levi-Civita symbol,

$$\epsilon_{\mu\nu\lambda\sigma} = \begin{cases} +1 & \text{if } \mu\nu\lambda\sigma \text{ is an even permutation of } 1234 \\ -1 & \text{if } \mu\nu\lambda\sigma \text{ is an odd permutation of } 1234 \\ 0 & \text{otherwise} \end{cases} \tag{7.2.34}$$

In terms of $\epsilon_{\mu\nu\lambda\sigma}$ one can show that,

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\,\epsilon_{\mu\nu\lambda\sigma}\,F_{\lambda\sigma} \tag{7.2.35}$$

One can show that the dual field strength tensor $\tilde{F}_{\mu\nu}$ is really a 2nd rank *psuedo* 4-tensor – it transforms under a Lorentz transformation just like a 2nd rank tensor, but it transforms with the wrong sign under a parity transformation.

From $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ we can construct two Lorentz invariant scalars (actually one scalar, and one psuedo-scalar). These are,

$$\frac{1}{2}F_{\mu\nu}F_{\mu\nu} = B^2 - E^2 \qquad \text{and} \qquad -\frac{1}{4}F_{\mu\nu}\tilde{F}_{\mu\nu} = \mathbf{B}\cdot\mathbf{E} \tag{7.2.36}$$

Thus $B^2 - E^2$ and $\mathbf{B}\cdot\mathbf{E}$ have the same value in all inertial frames of reference.

From the above we can conclude that if $\mathbf{E}\perp\mathbf{B}$ (so that $\mathbf{B}\cdot\mathbf{E} = 0$), and $|\mathbf{E}| = |\mathbf{B}|$ (so that $B^2 - E^2 = 0$) in one frame of reference, then this is also so in all inertial frames of reference. Thus the key features of electromagnetic waves in a vacuum, that $\mathbf{E}\perp\mathbf{B}$ and the fields have equal magnitudes, hold in all inertial frames of reference.

Also, if $\mathbf{E}\cdot\mathbf{B} = 0$ in one frame of reference, and $E^2 > B^2$ in that frame, then there exists a frame in which $\mathbf{B}' = 0$ and ${E'}^2 = E^2 - B^2$. And if $\mathbf{E}\cdot\mathbf{B} = 0$ in one frame and $B^2 > E^2$ in that frame, then there exists a frame in which $\mathbf{E}' = 0$

and ${B'}^2 = B^2 - E^2$. These conditions tell us when it is possible to make a Lorentz transformation and completely eliminate one of the fields $\mathbf{E}$ or $\mathbf{B}$.

Transformation Law for Electric and Magnetic Fields

From Eq. (7.2.17) we can derive the transformation laws for the electric and magnetic fields under a Lorentz transformation.

$$F'_{\mu\nu} = a_{\mu\sigma} a_{\nu\lambda} F_{\sigma\lambda} = a_{\mu\sigma} F_{\sigma\lambda} a^t_{\lambda\nu} \qquad \text{or more symbolically} \qquad [F'] = [a]\cdot[F]\cdot[a^t] \tag{7.2.37}$$

In the second way we have written the transformation law for $F_{\mu\nu}$ in the form of a matrix multiplication, in the same form as a similarity transformation for a 3×3 matrix under a rotation of the spatial coordinates.

For a transformation from inertial frame $\mathcal{K}$ to inertial frame $\mathcal{K}'$, where $\mathcal{K}'$ moves with velocity $v\hat{\mathbf{x}}$ as seen by $\mathcal{K}$, the above gives,

$$\begin{aligned} E'_1 &= E_1 & B'_1 &= B_1 \\ E'_2 &= \gamma\left(E_2 - \frac{v}{c}B_3\right) & B'_2 &= \gamma\left(B_2 + \frac{v}{c}E_3\right) \\ E'_3 &= \gamma\left(E_3 + \frac{v}{c}B_2\right) & B'_3 &= \gamma\left(B_3 - \frac{v}{c}E_2\right) \end{aligned} \tag{7.2.38}$$

Note that, unlike 4-vectors where it is the spatial component parallel to the direction of relative motion that contracts, while the directions transverse to the relative motion stay unchanged, for the fields $\mathbf{E}$ and $\mathbf{B}$ it is the components in the directions transverse to the relative motion that contract, while the component parallel to the relative motion stays unchanged. This is because the transformation law for $\mathbf{E}$ and $\mathbf{B}$ comes from the transformation law of the 2nd rand tensor $F_{\mu\nu}$, which is different from that of a 4-vector.