## Unit 7-3: Relativistic Kinematics

In this section we discuss how to formulate the equations of motion for particles in a relativistic way. This will lead us to a relativisitc understanding of momentum and energy.

### 4-Momentum

We already saw the 4-velocity $u_\mu$ of a particle. Now we define the 4-momentum,

$$p_\mu = m u_\mu = (m\gamma\mathbf{v}, imc\gamma) \qquad \textit{4-momentum} \tag{7.3.1}$$

where $m$ is the mass of the particle in the frame of reference in which the particle is at rest (the "rest mass"). The 4-momentum is a 4-vector since $u_\mu$ is a 4-vector and mass $m$ is a Lorentz invariant scalar. Remember, the speed $v$ that enters the factor $\gamma = 1/\sqrt{1-v^2/c^2}$ in the above expression is the ordinary velocity of the particle, $v = |d\mathbf{r}/dt|$.

The square of the 4-momentum is,

$$p_\mu^2 = m^2 u_\mu^2 = -m^2c^2 \tag{7.3.2}$$

where we used the result $u_\mu^2 = -c^2$ from Notes 7-1.

### 4-Force, or Minkowski Force

Now we want to specify the relativistic analog of force. This is the 4-force, or also called the Minkowski force. We will define it as,

$$K_\mu = (\mathbf{K}, iK_0) \qquad \textit{Minkowski force} \tag{7.3.3}$$

where we will have to determine how the spatial part $\mathbf{K}$ and the temporal part $iK_0$ are related to familiar quantities from Newtonian mechanics.

We will assume an analog of Newton's 2nd law of motion,

$$m\frac{d^2x_\mu}{ds^2} = K_\mu \qquad \text{recall, } ds \text{ is the proper time interval, } ds = dt/\gamma \tag{7.3.4}$$

With $dx_\mu/ds = u_s$, and $mu_\mu = p_\mu$, this gives,

$$m\frac{d^2x_\mu}{ds^2} = m\frac{du_\mu}{ds} = \boxed{\frac{dp_\mu}{ds} = K_\mu} \tag{7.3.5}$$

Now we had,

$$p_\mu^2 = -m^2c^2 \qquad \Rightarrow \qquad \frac{1}{2}\frac{d}{ds}(p_\mu^2) = p_\mu\frac{dp_\mu}{ds} = p_\mu K_\mu = 0 \qquad \text{since } p_\mu^2 \text{ is a constant} \tag{7.3.6}$$

$$\Rightarrow \qquad p_\mu K_\mu = m\gamma\mathbf{v}\cdot\mathbf{K} - mc\gamma K_0 = 0 \qquad \Rightarrow \qquad K_0 = \frac{\mathbf{v}}{c}\cdot\mathbf{K} \tag{7.3.7}$$

So once we have determined the spatial part $\mathbf{K}$, we will then know the temporal part $K_0$.

We will define the familiar Newtonian "3-force" $\mathbf{F}$ by,

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} \qquad \text{where the Newtonian momentum } \mathbf{p} \text{ is identified as the space components of } p_\mu \tag{7.3.8}$$

Then, from the spatial part of Eq. (7.3.5) and using $ds = dt/\gamma$, we have,

$$\mathbf{K} = \frac{d\mathbf{p}}{ds} = \gamma\frac{d\mathbf{p}}{dt} = \gamma\mathbf{F} \qquad \Rightarrow \qquad \boxed{\mathbf{K} = \gamma\mathbf{F} \qquad \text{and so} \qquad K_0 = \gamma\frac{\mathbf{v}}{c}\cdot\mathbf{F}} \tag{7.3.9}$$

**Kinetic Energy**

Consider now the 4th component of our relativistic Newton's equation (7.3.5),

$$m\frac{du_4}{ds} = m\frac{d(ic\gamma)}{ds} = iK_0 = i\gamma\frac{\mathbf{v}}{c}\cdot\mathbf{F} \quad \Rightarrow \quad md(\gamma) = d(m\gamma) = \gamma\frac{\mathbf{v}}{c^2}\cdot\mathbf{F}\,ds = \frac{\mathbf{v}}{c^2}\cdot\mathbf{F}\,dt = \frac{\mathbf{F}\cdot d\mathbf{r}}{c^2} \tag{7.3.10}$$

where $d\mathbf{r} = \mathbf{v}dt$. Now we know that $\mathbf{F}\cdot d\mathbf{r} = dW$, the work done by the force $\mathbf{F}$ in moving the particle $d\mathbf{r}$. So,

$$d(m\gamma c^2) = \mathbf{F}\cdot d\mathbf{r} = dW \tag{7.3.11}$$

So by the work-energy theorem of mechanics, we now associate $d(m\gamma c^2)$ as the change in the kinetic energy of the particle. And so we conclude,

$$\boxed{E = m\gamma c^2} \qquad \text{is the relativistic form for the particle's kinetic energy} \tag{7.3.12}$$

Comparing to the 4-momentum, $p_\mu = (m\gamma\mathbf{v},\ imc\gamma)$, we see that we can write the 4-momentum as,

$$\boxed{p_\mu = \left(\mathbf{p},\ \frac{iE}{c}\right)} \qquad \text{which is thus also called the } \textit{energy-momentum} \text{ 4-vector} \tag{7.3.13}$$

Written this way, and using Eq. (7.3.2) we have,

$$p_\mu^2 = p^2 - E^2/c^2 = -m^2c^2 \qquad \Rightarrow \qquad E^2 = (pc)^2 + (mc^2)^2 \tag{7.3.14}$$

So the temporal component of the 4-momentum is just $(i/c)$ times the relativistic kinetic energy of the particle. We thus have,

$$\boxed{\mathbf{p} = m\gamma\mathbf{v} \qquad \text{is the relativistic momentum;} \qquad E = m\gamma c^2 \qquad \text{is the relativistic kinetic energy}} \tag{7.3.15}$$

Now using our expression for $\gamma$ we have,

$$E = m\gamma c^2 = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{7.3.16}$$

For a particle moving non-relativistically, with $v \ll c$, we can expand the square root to get,

$$E \simeq mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) = mc^2 + \frac{1}{2}mv^2 \tag{7.3.17}$$

The second term on the right hand side is just the usual Newtonian kinetic energy, $\frac{1}{2}mv^2$. The first term, $mc^2$, is referred to as the "rest mass energy."

For non-relativistic particles, if particles are neither created nor destroyed, and each particle $i$ that enters an interaction leaves with the same mass $m_i$ that it entered with, then the rest mass before the interaction will equal the rest mass after the interaction, and all the interesting effects required by energy conservation will be governed by the non-relativistic kinetic energy. However, when particles change their nature, such as in the nuclear fission reaction

$$^{236}\text{U} \quad \rightarrow \quad ^{92}\text{Kr} \ + \ ^{141}\text{Ba} \tag{7.3.18}$$

then considerable rest mass energy can be involved.

We thus see that our relativistic Newton's equation of motion,

$$\frac{dp_\mu}{ds} = K_\mu \tag{7.3.19}$$

is not only the analog of Newton's 2nd law of motion (which comes from the spatial components) but also gives the work-energy theorem (from the temporal component).

The above results are very aesthetically pleasing! But you might ask, how do we know that the correct thing to do is to define the ordinary Newtonian momentum as the spatial parts of the 4-momentum $p_\mu$, i.e. take $\mathbf{p} = m\gamma\mathbf{v}$? Why shouldn't we keep momentum as the familiar $m\mathbf{v}$? The answer has to do with momentum conservation. If momentum is conserved in one inertial frame of reference, we would want it to be conserved in *all* inertial frames of reference (otherwise, we could distinguish between different inertial frames of reference, and that would violate one of the fundamental assumptions of special relativity).

Suppose we used the familiar $m\mathbf{v}$ as the momentum. If momentum is conserved for some collection of particles, we would then have,

$$\sum_i m_i\mathbf{v}_i(t_1) = \sum_i m_i\mathbf{v}_i(t_2) \qquad \text{for any times } t_1 \text{ and } t_2 \tag{7.3.20}$$

Note, the individual particle velocities $\mathbf{v}_i$ are not generally constant, but the above sum must be.

Now if the above holds in inertial frame $\mathcal{K}$, and we now transform to another frame $\mathcal{K}'$ moving with velocity $\mathbf{w}$ with respect to $\mathcal{K}$, then we can apply the transformation laws for velocity (see Griffiths example 12.6, for example) to compute the velocities $\mathbf{v}'_i$ of each of the particles in inertial frame $\mathcal{K}'$. We would then generally find that,

$$\sum_i m_i\mathbf{v}'_i(t'_1) \neq \sum_i m_i\mathbf{v}'_i(t'_2) \qquad \text{for any times } t'_1 \text{ and } t'_2 \tag{7.3.21}$$

This results from the complicated transformation laws for velocities, where $\mathbf{v}'_i$ will depend on how much of $\mathbf{v}_i$ is oriented parallel to $\mathbf{w}$ and how much of $\mathbf{v}_i$ is oriented perpendicular to $\mathbf{w}$. Thus momentum would not generally be conserved in the frame $\mathcal{K}'$.

However if we use $\mathbf{p}_i = m_i\gamma_i\mathbf{v}_i$ as the momentum of particle $i$, then momentum conservation would be given by the spatial components of,

$$p_\mu^{\text{total}}(t_1) = \sum_i p_{\mu i}(t_1) = \sum_i p_{\mu i}(t_2) = p_\mu^{\text{total}}(t_2) \qquad \text{for any times } t_1 \text{ and } t_2 \tag{7.3.22}$$

The temporal component of the above gives energy conservation!

Now if the above holds in the inertial frame $\mathcal{K}$, then when we transform to the frame $\mathcal{K}'$, both $p_\mu^{\text{total}}(t_1)$ and $p_\mu^{\text{total}}(t_2)$ will transform exactly the same way since they are both 4-vectors. Thus momentum *and* energy conservation will hold also in $\mathcal{K}'$.

Thus we see that $\mathbf{p} = m\gamma\mathbf{v}$ is intimately related to momentum conservation. The fact that kinetic energy $E$ is given by the temporal component of the energy-momentum 4-vector $p_\mu$ shows that there is a deep connection between conservation of energy and conservation of momentum.

**The Lorentz Force**

We have as the equation of motion of a particle,

$$\frac{dp_\mu}{ds} = K_\mu \tag{7.3.23}$$

We now ask, for a charged particle, what is the $K_\mu$ that represents the Lorentz force? And how can we write it in a Lorentz invariant way? Will there be any relativistic corrections to $\mathbf{F} = q\mathbf{E} + q(\mathbf{v}/c)\times\mathbf{B}$?

$K_\mu$ should depend on the fields $F_{\mu\nu}$ and the charge's trajectory $x_\mu(s)$. As $\mathbf{v}\to 0$, we must have $\mathbf{K} = \gamma\mathbf{F} \to \mathbf{F} = q\mathbf{E}$.

$K_\mu$ can't depend explicitly on $x_\mu$, as it should be independent of where we put the origin of our coordinates. So $K_\mu$ can depend on the charge's trajectory only via the derivatives, $\dot{x}_\mu$, $\ddot{x}_\mu$, $\dddot{x}_\mu$, etc.

As $\mathbf{v} \to 0$, we know that $\mathbf{K} \to q\mathbf{E}$ does not depend on the acceleration or higher time derivatives of the charge's motion, so $K_\mu$ cannot depend on $\ddot{x}_\mu$, $\dddot{x}_\mu$, or higher derivatives.

Thus $K_\mu$ can depend only on $F_{\mu\nu}$ and $\dot{x}_\mu$. We need to form a 4-vector out of these quantities that is *linear* in the fields $F_{\mu\nu}$ and is proportional to the charge $q$. The only possibility is,

$$q\, f(\dot{x}_\mu^2)\, F_{\mu\nu}\, \dot{x}_\nu \qquad \text{where } f(\cdot) \text{ is an as yet unknown function.} \tag{7.3.24}$$

But $\dot{x}_\mu^2 = u_\mu^2 = -c^2$ is a constant. So $f(\dot{x}_\mu^2)$ is a constant, and we will choose it to be $1/c$. This gives,

$$\boxed{K_\mu = \frac{q}{c} F_{\mu\nu} \dot{x}_\nu} \qquad \text{as the only possibility to represent the Lorentz force.} \tag{7.3.25}$$

This would then give for the force,

$$\mathbf{F} = \frac{1}{\gamma}\mathbf{K} \tag{7.3.26}$$

For the $i$th spatial component we have,

$$F_i = \frac{1}{\gamma} K_i = \frac{q}{\gamma c}\left(F_{ij}\dot{x}_j + F_{i4}\dot{x}_4\right) = \frac{q}{\gamma c}\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right)\dot{x}_j + \frac{q}{\gamma c}(-iE_i)(ic\gamma) \tag{7.3.27}$$

Now, noting that the term involving the derivatives of the $A_i$ give the magnetic field, and using for the spatial components of the 4-velocity $\dot{x}_j = \gamma v_j$, we have,

$$F_i = \frac{q}{\gamma c}\left(\epsilon_{ijk} B_k\, \gamma\, v_j\right) + \frac{q}{\gamma c} E_i\, c\, \gamma = qE_i + q\, \epsilon_{ijk} \frac{v_j}{c} B_k \tag{7.3.28}$$

or in vector form,

$$\mathbf{F} = q\mathbf{E} + q\,\frac{\mathbf{v}}{c} \times \mathbf{B} \tag{7.3.29}$$

Thus the Lorentz force is exactly given by $\mathbf{K}/\gamma$ with $K_\mu$ as in Eq. (7.3.25).

There are no relativistic corrections to the Lorentz force, and the Lorentz force has exactly the same form in all inertial frames of reference!