transform variables to $x = \beta\epsilon$.

then we want to do integrals of the form

$$\Phi \equiv \int_0^\infty dx \, \frac{\phi(x)}{z^{-1}e^{x}+1}$$

$\phi(x)$ is any function of x.

For example, to get the "standard" function $f_n(z)$, we use $\phi(x) = \frac{1}{\Gamma(n)} x^{n-1}$

Define $\xi = \beta\mu = \ln z$

$$\Phi = \int_0^\infty dx \, \frac{\phi(x)}{e^{x-\xi}+1}$$

Define $\psi(x) \equiv \int_0^x \phi(x')\,dx'$, $f(x) \equiv \frac{1}{[e^{x-\xi}+1]}$ fermi function

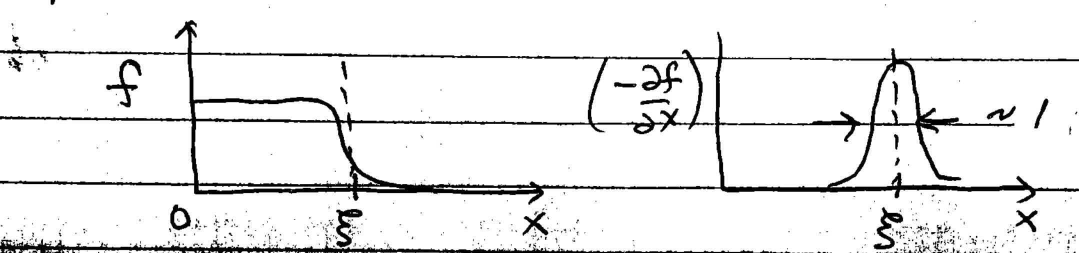
$$\Phi = \int_0^\infty dx \left(\frac{\partial\psi}{\partial x}\right) f(x)$$ integrate by parts

$$= \psi(x) f(x) \Big|_0^\infty + \int_0^\infty dx \, \psi(x) \left(-\frac{\partial f}{\partial x}\right)$$

$$= \int_0^\infty dx \, \psi(x) \left(-\frac{\partial f}{\partial x}\right)$$ since $\psi(0) = 0$ and $f(\infty) = 0$

1st term vanishes

Now we use the fact that at low T, $\left(-\frac{\partial f}{\partial x}\right)$ is strongly peaked about $x = \xi$



$\xi \gtrsim 1$

$\xi \sim \frac{\epsilon_F}{kT}$ large

expand $\psi(x)$ about $x = \xi$

$$\psi(x) = \sum_{n=0}^{\infty} \left.\frac{d^n\psi}{dx^n}\right|_{x=\xi} \frac{(x-\xi)^n}{n!}$$

$$\Rightarrow \quad \Phi = \sum_{n=0}^{\infty} \left.\frac{d^n\psi}{dx^n}\right|_{x=\xi} \int_0^{\infty} dx \, \frac{(x-\xi)^n}{n!} \left(-\frac{\partial f}{\partial x}\right)$$

since $\left(-\frac{\partial f}{\partial x}\right)$ is zero except for a region of order 1 about $x = \xi \gg 1$, we can replace the lower limit of the integral by $-\infty$ without any noticeable change

Then we can make a change of variables $y = x - \xi$ and the integrals become

$$\int_0^{\infty} dy \, \frac{y^n}{n!} \left(\frac{-\partial f}{\partial y}\right) \qquad \text{where } f(y) = \frac{1}{e^y + 1}$$

Now $$-\frac{\partial f}{\partial y} = \frac{e^y}{(e^y+1)^2} = \frac{e^y}{e^{2y} + 2e^y + 1} = \frac{1}{e^y + 2 + e^{-y}}$$

is <u>symmetric</u> about $y = 0$.

$\Rightarrow$ all the integrals for n odd vanish!

To sum over only n even terms, let $n \to 2n$

$$\Phi = \sum_{n=0}^{\infty} \frac{d^{2n}\psi}{dx^{2n}}\Big|_{x=\xi} \int_{-\infty}^{\infty} dy \, \frac{y^{2n}}{(2n)!} \left(-\frac{\partial f}{\partial y}\right)$$

let $a_n \equiv \int_{-\infty}^{\infty} dy \, \frac{y^{2n}}{(2n)!}\left(-\frac{\partial f}{\partial y}\right)$, $a_0 = \int_{-\infty}^{\infty} dy \left(-\frac{\partial f}{\partial y}\right) = 1$

The a_n are just numbers that be computed.
They contain no system parameters whatsoever

For $n \geq 1$ one can show

$$a_n = 2\left(1 - \frac{1}{2^{2n}} + \frac{1}{3^{2n}} - \frac{1}{4^{2n}} + \frac{1}{5^{2n}} - \cdots\right)$$

$$= \left(2 - \frac{1}{2^{2(n-1)}}\right) \zeta(2n)$$

where $\zeta(n) = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots$ is the Riemann zeta function

In particular $a_1 = \frac{\pi^2}{6}$, $a_2 = \frac{7\pi^4}{360}$

$$\Phi = \sum_{n=0}^{\infty} a_n \frac{d^{2n}\psi}{dx^{2n}}\Big|_{x=\xi} = \psi(\xi) + \sum_{n=1}^{\infty} a_n \frac{d^{2n}\psi}{dx^{2n}}\Big|_{x=\xi}$$

use $\frac{d\psi}{dx} = \phi$ to finally get $\qquad \psi(x) = \int_0^x dx' \phi(x')$

$$\Phi = \int_0^{\xi} dx\, \phi(x) + \sum_{n=1}^{\infty} a_n \frac{d^{2n-1}\phi}{dx^{2n-1}}\Big|_{x=\xi}$$

$$= \int_0^{\xi} dx\, \phi(x) + \frac{\pi^2}{6}\frac{d\phi}{dx}\Big|_{x=\xi} + \frac{7\pi^4}{360}\frac{d^3\phi}{dx^3}\Big|_{x=\xi} + \cdots$$

This gives a power series in temperature.

To see this, transform back to the energy variable

$x = \beta\epsilon \quad , \quad \epsilon = k_B T x$

$$\Phi \equiv \int_0^\infty d\epsilon \frac{\phi(\epsilon)}{z^{-1}e^{\beta\epsilon}+1} = k_B T \left\{ \int_0^\infty dx \frac{\phi(k_B T x)}{z^{-1}e^{x}+1} \right\}$$

using $$k_B T \int_0^{\xi = \mu/k_B T} dx\, \phi(k_B T x) = \int_0^\mu d\epsilon\, \phi(\epsilon)$$

and $$\frac{d\phi}{dx} = \frac{d\phi}{d\epsilon}\frac{d\epsilon}{dx} = \frac{d\phi}{d\epsilon} k_B T$$

we get

$$\Phi = \int_0^\infty d\epsilon\, \phi(\epsilon) n(\epsilon)$$

$$\boxed{\Phi = \int_0^\mu d\epsilon\, \phi(\epsilon) + \frac{\pi^2}{6}(k_B T)^2 \frac{d\phi}{d\epsilon}\Big|_{\epsilon=\mu} + \frac{7\pi^4}{360}(k_B T)^4 \frac{d^3\phi}{d\epsilon^3}\Big|_{\epsilon=\mu} + \cdots}$$

<u>Example</u>

① density $n = \frac{N}{V} = \int_0^\infty d\epsilon\, g(\epsilon)\, n(\epsilon) \quad \Rightarrow \phi(\epsilon) \equiv g(\epsilon)$

$$n = \int_0^\mu d\epsilon\, g(\epsilon) + \frac{\pi^2}{6}(k_B T)^2 \frac{dg}{d\epsilon}\Big|_{\epsilon=\mu} + \cdots$$

Now as $T \to 0$, $\mu \to \epsilon_F$ the fermi energy

$$n = \int_0^{\epsilon_F} d\epsilon\, g(\epsilon) + \int_{\epsilon_F}^{\mu} d\epsilon\, g(\epsilon) + \frac{\pi^2}{6}(k_B T)^2 \left.\frac{dg}{d\epsilon}\right|_{\epsilon=\mu}$$

But ϵ_F was determined by $n = \int_0^{\epsilon_F} d\epsilon\, g(\epsilon)$

$$\Rightarrow \int_{\epsilon_F}^{\mu} d\epsilon\, g(\epsilon) = \frac{-\pi^2}{6}(k_B T)^2 \left.\frac{dg}{d\epsilon}\right|_{\epsilon=\mu}$$

since left hand side is $O(kT)^2$ is small, we can approx ~~the right hand side as~~ it as

$$\int_{\epsilon_F}^{\mu} d\epsilon\, g(\epsilon) \approx (\mu - \epsilon_F)\, g(\epsilon_F)$$

$$\Rightarrow (\mu - \epsilon_F) \simeq \frac{-\pi^2}{6} \frac{(k_B T)^2}{g(\epsilon_F)} \left.\frac{dg}{d\epsilon}\right|_{\epsilon=\mu}$$

so $\mu - \epsilon_F \sim O(k_B T)^2$ is small, so to lowest order can evaluate $\frac{dg}{d\epsilon}$ on right hand side at $\epsilon = \epsilon_F$ instead of $\epsilon = \mu$

$$\boxed{\mu(T) \simeq \epsilon_F - \frac{\pi^2}{6}(k_B T)^2 \frac{g'(\epsilon_F)}{g(\epsilon_F)}} \qquad g' = \frac{dg}{d\epsilon}$$

Shows that chemical potential μ decreases from ϵ_F by $O(kT)^2$ at low T

For free electrons where $g(\epsilon) = C\sqrt{\epsilon}$

$$g'(\epsilon) = \tfrac{1}{2} C \frac{1}{\sqrt{\epsilon}}$$

$$\mu(T) \simeq \epsilon_F - \frac{\pi^2}{6}(k_B T)^2 \frac{1}{2\epsilon_F} = \epsilon_F - \frac{\pi^2}{12}\frac{(k_B T)^2}{\epsilon_F}$$

$$\boxed{\mu(T) \simeq \epsilon_F\left(1 - \frac{1}{3}\left(\frac{\pi k_B T}{2\epsilon_F}\right)^2\right) = \epsilon_F\left(1 - \frac{1}{3}\left(\frac{\pi}{2}\frac{T}{T_F}\right)^2\right)}$$

correction is small for metals at room temp as $T_F \sim 10{,}000\,°K$

(2) energy $\frac{E}{V} = \int_0^\infty d\epsilon\, g(\epsilon)\, \epsilon\, n(\epsilon) \Rightarrow \phi(\epsilon) = g(\epsilon)\epsilon$

$$u = \frac{E}{V} = \int_0^\mu d\epsilon\, g(\epsilon)\epsilon + \frac{\pi^2}{6}(k_B T)^2\left[g(\mu) + \mu g'(\mu)\right]$$

$$= \underbrace{\int_0^{\epsilon_F} d\epsilon\, g(\epsilon)\epsilon}_{= u(0)} + \underbrace{\int_{\epsilon_F}^{\mu} d\epsilon\, g(\epsilon)\epsilon}_{\simeq (\mu-\epsilon_F) g(\epsilon_F)\epsilon_F} + \frac{\pi^2 (k_B T)^2}{6}\left[g(\mu) + \mu g'(\mu)\right]$$

$= u(0)$ ground state energy density; $\simeq (\mu - \epsilon_F) g(\epsilon_F)\epsilon_F$ as before; replace $\mu \approx \epsilon_F$ as before

$$u(T) = u(0) + (\mu - \epsilon_F) g(\epsilon_F)\epsilon_F + \frac{\pi^2 (k_B T)^2}{6}\left[g(\epsilon_F) + \epsilon_F g'(\epsilon_F)\right]$$

$$= u(0) + \left[-\frac{\pi^2}{6}(k_B T)^2 \frac{g'(\epsilon_F)}{g(\epsilon_F)}\right] g(\epsilon_F)\epsilon_F + \frac{\pi^2 (k_B T)^2}{6}\left[g(\epsilon_F) + \epsilon_F g'(\epsilon_F)\right]$$

$$\boxed{u(T) = u(0) + \frac{\pi^2 (k_B T)^2}{6} g(\epsilon_F)}$$

specific heat per volume

$$c_V \equiv \frac{C_V}{V} = \frac{1}{V}\left(\frac{dE}{dT}\right)_V = \left(\frac{du}{dT}\right)_V$$

$$\boxed{c_V = \frac{\pi^2}{3} k_B^2 T\, g(\epsilon_F)}$$

for free electrons we can write $g(\epsilon) = C\sqrt{\epsilon}$

$$n = \int_0^{\epsilon_F} d\epsilon\, g(\epsilon) = \frac{2}{3} C\, \epsilon^{3/2} \Rightarrow C = \frac{3}{2}\frac{n}{\epsilon^{3/2}}$$

$$\Rightarrow g(\epsilon_F) = \frac{3}{2}\frac{n}{\epsilon_F^{3/2}} \cdot \epsilon_F^{1/2} = \frac{3}{2}\frac{n}{\epsilon_F}$$ density of states at fermi energy

$$\boxed{c_V = \frac{\pi^2}{2}\left(\frac{k_B T}{\epsilon_F}\right) n k_B}$$

or total specific heat $C_V = V c_V$ $\qquad nV = N$

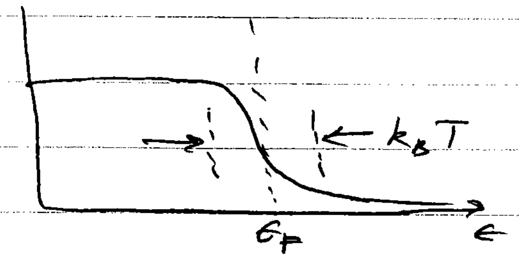
$$C_V = \frac{\pi^2}{2}\left(\frac{k_B T}{\epsilon_F}\right) N k_B$$

$\Rightarrow$ specific heat due to fermi gas of electrons in a conductor is $C_V \sim T$ at low temperatures

We already saw that specific heat due to ionic vibrations (phonons) in a solid went like $C_V \sim T^3$ at low temperatures (Debye model)

$\Rightarrow$ electronic contribution to C_V dominates at sufficiently low T.

Simple estimate of C_V



When increase temperature to $k_B T$, the electrons near the fermi energy ϵ_F will increase their energy by an amount $\sim k_B T$. The number of such electrons ~~is roughly~~ per unit volume is roughly

$$g(\epsilon_F)(k_B T)$$

density of states at ϵ_F ; energy interval about ϵ_F of states which ~~increase~~ get excited

$\Rightarrow$ increase in energy per unit volume is

$$\Delta u \sim \underbrace{\left(g(\epsilon_F)\, k_B T\right)}_{\#\text{ electrons excited}} \underbrace{(k_B T)}_{\text{excitation energy per excited electron}} \sim g(\epsilon_F)(k_B T)^2$$

$$\Rightarrow C_V = \frac{d\Delta u}{dT} \sim g(\epsilon_F)\, k_B^2\, T$$

The previous calculation gives the precise numerical coefficient electronic specific heat per volume

$$C_V^{elec} = \frac{\pi^2}{2}\left(\frac{k_B T}{\epsilon_F}\right)\frac{N}{V}k_B \left(1 + O\left(\frac{k_B T}{\epsilon_F}\right)^2\right)$$

compare to classical result $C_V^{classical} = \frac{N}{V}k_B$

The correct result for degenerate fermi gas is a factor $\frac{\pi^2}{2}\left(\frac{k_B T}{\epsilon_F}\right) = \frac{\pi^2}{2}\left(\frac{T}{T_F}\right)$ smaller than classical result by factor $\sim \frac{10^2}{10^4} = 10^{-2}$ at room temperature

also, classical C_V is indep of T, whereas fermi gas result is $\propto T$.

At low T, the ionic contribution to C_V is

$$C_V^{ion} = \frac{12\pi^4}{5}\left(\frac{T}{\Theta_D}\right)^3 \frac{N}{V}k_B$$

$$\frac{C^{elec}}{C^{ion}} = \frac{\pi^2}{2}\left(\frac{k_B T}{\epsilon_F}\right)\frac{5}{12\pi^4}\left(\frac{\Theta_D}{T}\right)^3 \simeq \frac{5}{24\pi^2}\left(\frac{\Theta_D}{T_F}\right)\left(\frac{\Theta_D}{T}\right)^2$$

$$\simeq 1 \quad \text{when} \quad T^* = \sqrt{\frac{5}{24\pi^2}\left(\frac{\Theta_D}{T_F}\right)}\,\Theta_D \simeq 0.15\left(\frac{\Theta_D}{T_F}\right)^{1/2}\Theta_D$$

for metals, $T_F \sim 10^4\,°K$, $\Theta_D \sim 10^2\,°K$

$$T^* = 0.15\sqrt{10^{-2}}\,\Theta_D \simeq 0.015\,\Theta_D$$

So ionic contrib to C_V dominates over electronic contrib until $T \lesssim 0.01\,\Theta_D$ ie at $O(1)\,°K$. The electronic contrib dominates at lower temperatures.