integrate by parts

$$\frac{pV}{k_B T} = -\frac{V}{\pi^2 c^3}\left[\frac{\omega^3}{3}\ln\left(1-e^{-\beta\hbar\omega}\right)\right]_0^\infty + \frac{V}{\pi^2 c^3}\int d\omega\, \frac{\omega^3}{3}\, \frac{\beta\hbar e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}}$$

$$\underset{0}{\shortparallel}$$

$$\frac{pV}{k_B T} = \frac{V\beta\hbar}{3\pi^2 c^3}\int_0^\infty d\omega \left(\frac{\omega^3}{e^{\beta\hbar\omega}-1}\right)$$

compare with computation of $\frac{U}{V}$

$$= \frac{\beta}{3} U = \frac{1}{3}\frac{U}{k_B T}$$

$$\Rightarrow \boxed{\frac{1}{3}U = PV}$$ pressure of photon gas

compare to non relativistic ideal gas

$U = \frac{3}{2} N k_B T$, $PV = N k_B T \Rightarrow \frac{2}{3} U = PV$

<u>Ideal Quantum Gas</u> – Grand canonical ensemble

$$\ln \mathcal{Z} = \pm \sum_i \ln\left(1 \pm e^{-\beta(\epsilon_i - \mu)}\right) \qquad +\text{ FD}, \ -\text{ BE}$$

<u>for free particles</u>, states can be labeled by ~~wavevector~~ wavevector $\vec{k}$ with $k_\mu = \frac{2\pi n_\mu}{L}$, $n_\mu = 0, 1, \ldots$ due to periodic boundary conditions. volume $V = L^3$

$$\Rightarrow \underset{\text{states}}{\sum_i} \rightarrow \underbrace{\sum_s}_{\text{spin polarizations}} \sum_{\vec{k}} \rightarrow \underbrace{g_s}_{\#\text{ spin states for each } \vec{k}} \frac{V}{(2\pi)^3} \int_0^\infty dk\, 4\pi k^2$$

for free particles, ϵ depends only on $|\vec{k}|$. Define <u>density of states</u> $g(\epsilon)$ such that

$$\frac{g_s}{(2\pi)^3} \int dk\, 4\pi k^2 = \int g(\epsilon)\, d\epsilon$$

$g(\epsilon)$ = # states with energy ϵ per unit energy per volume

$$\Rightarrow g(\epsilon) = \frac{g_s 4\pi}{(2\pi)^3} k^2 \frac{dk}{d\epsilon}$$

For <u>non-relativistic particles</u> $\epsilon = \frac{\hbar^2 k^2}{2m}$, $k = \sqrt{\frac{2m\epsilon}{\hbar^2}}$

$$g(\epsilon) = \frac{g_s 4\pi}{(2\pi)^3} \frac{2m\epsilon}{\hbar^2} \sqrt{\frac{2m}{\hbar^2}} \frac{1}{2\sqrt{\epsilon}}$$

$$= \frac{2\pi g_s}{(2\pi)^3} \left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{\epsilon} = \left(\frac{2\pi m}{\hbar^2}\right)^{3/2} \frac{2^{3/2} (2\pi)^{3/2}}{(2\pi)^2} g_s \sqrt{\epsilon}$$

<u>Density of States</u>

$$\boxed{g(\epsilon) = \left(\frac{2\pi m}{h^2}\right)^{3/2} \frac{2 g_s}{\sqrt{\pi}} \sqrt{\epsilon}}$$

$g \sim \sqrt{\epsilon}$

pressure

$$\frac{p}{k_B T} = \frac{1}{V} \ln \mathcal{L} = \pm \frac{1}{V} \sum_i \ln\left(1 \pm z e^{-\beta \epsilon_i}\right) \qquad z = e^{\beta\mu}$$

$$= \pm \int_0^\infty d\epsilon \; g(\epsilon) \ln\left(1 \pm z e^{-\beta\epsilon}\right)$$

$$= \pm \left(\frac{2\pi m}{h^2}\right)^{3/2} \frac{2 g_s}{\sqrt{\pi}} \int_0^\infty d\epsilon \sqrt{\epsilon} \ln\left(1 \pm z e^{-\beta\epsilon}\right)$$

substitute variables $y = \beta\epsilon$

$$\frac{p}{k_B T} = \pm \left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} \frac{2 g_s}{\sqrt{\pi}} \int_0^\infty dy \, y^{1/2} \ln\left(1 \pm z e^{-y}\right)$$

integrate by parts $\qquad \lambda = \left(\frac{h^2}{2\pi m k_B T}\right)^{1/2}$ thermal wavelength

$$\frac{p}{k_B T} = \pm \frac{2 g_s}{\sqrt{\pi}\,\lambda^3} \left\{ \underbrace{\frac{2}{3} y^{3/2} \ln\left(1 \pm z e^{-y}\right)\Big|_0^\infty}_{=0} - \int_0^\infty dy \, \frac{2}{3} y^{3/2} \frac{(\mp z e^{-y})}{1 \pm z e^{-y}} \right\}$$

$$\boxed{\frac{p}{k_B T} = \frac{4 g_s}{3\sqrt{\pi}\,\lambda^3} \int_0^\infty dy \, \frac{y^{3/2}}{z^{-1} e^{y} \pm 1}} \qquad \begin{matrix} + \text{FD} \\ - \text{BE} \end{matrix}$$

density of particles $\quad \frac{N}{V} = \sum_i \langle n_i \rangle$

$$\frac{N}{V} = \frac{1}{V} \sum_i \frac{1}{z^{-1} e^{\beta\epsilon_i} \pm 1} = \int_0^\infty d\epsilon \; g(\epsilon) \frac{1}{z^{-1} e^{\beta\epsilon} \pm 1}$$

$$= \left(\frac{2\pi m}{h^2}\right)^{3/2} \frac{2 g_s}{\sqrt{\pi}} \int_0^\infty d\epsilon \, \frac{\sqrt{\epsilon}}{z^{-1} e^{\beta\epsilon} \pm 1}$$

$$= \left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} \frac{2 g_s}{\sqrt{\pi}} \int dy \, \frac{y^{1/2}}{z^{-1} e^{y} \pm 1}$$

$$\boxed{\frac{N}{V} = \frac{2 g_s}{\sqrt{\pi}\,\lambda^3} \int_0^\infty dy \, \frac{y^{1/2}}{z^{-1} e^{y} \pm 1}} \qquad \begin{matrix} + \text{FD} \\ - \text{BE} \end{matrix}$$

energy density $\quad E = \sum_i \epsilon_i \langle n_i \rangle$

$$\frac{E}{V} = \frac{1}{V} \sum_i \frac{\epsilon_i}{z^{-1} e^{\beta\epsilon_i} \pm 1} = \int_0^\infty d\epsilon \, g(\epsilon) \frac{\epsilon}{z^{-1} e^{\beta\epsilon} \pm 1}$$

$$= \frac{2 g_s}{\sqrt{\pi}\,\lambda^3} k_B T \int_0^\infty dy \frac{y^{3/2}}{z^{-1} e^{y} \pm 1}$$

$$\frac{E}{V} = \frac{3}{2} k_B T \frac{4 g_s}{3\sqrt{\pi}\,\lambda^3} \int_0^\infty \frac{y^{3/2}}{z^{-1} e^{y} \pm 1} = \left(\frac{3}{2} k_B T\right)\left(\frac{p}{k_B T}\right)$$

$$\Rightarrow \frac{E}{V} = \frac{3}{2} p \quad \text{or} \quad \boxed{p = \frac{2}{3}\frac{E}{V}}$$ both fermions and bosons

Define "standard functions" (see Pathria Appendices D and E)

$$f_n(z) \equiv \frac{1}{\Gamma(n)} \int_0^\infty dy \frac{y^{n-1}}{z^{-1} e^{y} + 1} = \sum_{\ell=1}^\infty (-1)^{\ell+1} \frac{z^\ell}{\ell^n}$$

$$g_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty dy \frac{y^{n-1}}{z^{-1} e^{y} - 1} = \sum_{\ell=1}^\infty \frac{z^\ell}{\ell^n}$$

$$\Gamma(n+1) = n\,\Gamma(n)$$
$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$
$$\Rightarrow \Gamma(3/2) = \tfrac{1}{2}\sqrt{\pi}$$
$$\Gamma(5/2) = \tfrac{3}{4}\sqrt{\pi}$$

In terms of these:

	Fermions	Bosons
	$\frac{p}{k_B T} = \frac{g_s}{\lambda^3} f_{5/2}(z)$	$\frac{p}{k_B T} = \frac{g_s}{\lambda^3} g_{5/2}(z)$
	$\frac{N}{V} = \frac{g_s}{\lambda^3} f_{3/2}(z)$	$\frac{N}{V} = \frac{g_s}{\lambda^3} g_{3/2}(z)$
	$\frac{E}{V} = \frac{3}{2} k_B T \frac{g_s}{\lambda^3} f_{5/2}(z)$	$\frac{E}{V} = \frac{3}{2} k_B T \frac{g_s}{\lambda^3} g_{5/2}(z)$
	$\frac{E}{N} = \frac{3}{2} k_B T \frac{f_{5/2}(z)}{f_{3/2}(z)}$	$\frac{E}{N} = \frac{3}{2} k_B T \frac{g_{5/2}(z)}{g_{3/2}(z)}$

Equation of state: low densities - virial expansion

$z \ll 1$ "non-degenerate"

keep lowest terms in series expansion

$$\frac{p}{k_BT} = \frac{g_s}{\lambda^3}\begin{Bmatrix} f_{5/2} \\ g_{5/2}\end{Bmatrix} = \frac{g_s}{\lambda^3}\left(z \mp \frac{z^2}{2^{5/2}} + \cdots\right) \quad \begin{matrix} - \text{ FD} \\ + \text{ BE}\end{matrix}$$

$$\frac{N}{V} = \frac{g_s}{\lambda^3}\begin{Bmatrix} f_{3/2} \\ g_{3/2}\end{Bmatrix} = \frac{g_s}{\lambda^3}\left(z \mp \frac{z^2}{2^{3/2}} + \cdots\right)$$

$$\Rightarrow \frac{p}{k_BT} = \frac{N}{V}\frac{\left(z \mp \frac{z^2}{2^{5/2}} + \cdots\right)}{\left(z \mp \frac{z^2}{2^{3/2}} + \cdots\right)} = \frac{N}{V}\left(1 \mp \frac{z}{2^{5/2}} + \cdots\right)\left(1 \pm \frac{z}{2^{3/2}} + \cdots\right)$$

$$= \frac{N}{V}\left(1 \pm \frac{z}{2^{3/2}} \mp \frac{z}{2^{5/2}} + \cdots\right) \qquad \frac{1}{2^{3/2}} - \frac{1}{2^{5/2}} = \frac{2}{2^{5/2}} - \frac{1}{2^{5/2}} = \frac{1}{2^{5/2}}$$

$$pV = Nk_BT\left(1 \pm \frac{z}{2^{5/2}} + \cdots\right)$$

quantum correction to classical ideal gas law.

+ FD - p increases compared to classically
- effective repulsion due to Pauli exclusion

− BE - p decreases compared to classically
- effective attraction.

above is similar conclusion to what we saw from 2-particle density matrix.

for small z, the leading term gives $\frac{N}{V} = \frac{g_s}{\lambda^3} z$

or $z = \left(\frac{N}{V}\frac{\lambda^3}{g_s}\right)$ - same result we had classically

$\Rightarrow$ small z limit is the low density limit $n\lambda^3 \ll 1$

or high T

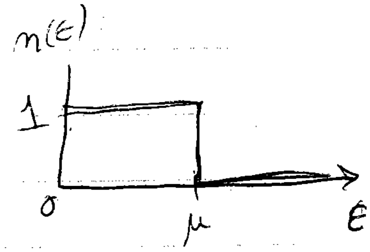
$$pV = Nk_BT\left(1 \pm \frac{1}{2^{5/2} g_s}\frac{N}{V}\lambda^3 + \cdots\right)$$

Sommerfeld model of electrons in a conductor

Fermi gas - high density / low temperature limit
"degenerate" fermi gas

Consider first $T \to 0$

$$\langle n(\epsilon) \rangle = \frac{1}{e^{\beta(\epsilon-\mu)}+1}$$



as $T \to 0$ $\quad e^{\beta(\epsilon-\mu)} \to \begin{cases} \infty & \epsilon > \mu \\ 0 & \epsilon < \mu \end{cases}$

$$\Rightarrow \langle n(\epsilon) \rangle \to \begin{cases} 0 & \epsilon > \mu \\ 1 & \epsilon < \mu \end{cases}$$

$\Rightarrow$ all states with $\epsilon < \mu$ are filled, all states with $\epsilon > \mu$ are empty. this is the $T=0$ ground state of the Fermi gas. We therefore see that $\mu(T=0)$ is the energy of the highest energy single particle state that is occupied in the ground state. One calls this energy the Fermi-energy

$$\epsilon_F \equiv \mu(T=0)$$

at $T=0$

$$N = g_s \sum_{\vec{k}} 1 \quad \text{count occupied states}$$

$\leftarrow$ s.t. $\frac{\hbar^2 k^2}{2m} \leq \epsilon_F$

$$= g_s \frac{V}{(2\pi)^3} 4\pi \int_0^{k_F} dk\, k^2 = \frac{g_s V}{6\pi^2} k_F^3 \quad \text{where } \frac{\hbar^2 k_F^2}{2m} = \epsilon_F$$

$$n \equiv \frac{N}{V} = \frac{g_s}{6\pi^2} k_F^3 = \frac{g_s}{6\pi^2}\left(\frac{2m\epsilon_F}{\hbar^2}\right)^{3/2}$$

or $\quad \epsilon_F = \frac{\hbar^2}{2m}\left(\frac{6\pi^2 n}{g_s}\right)^{2/3}, \quad k_F = \left(\frac{6\pi^2 n}{g_s}\right)^{1/3}$

relation between $\mu(T=0)$ and density $n = N/V$