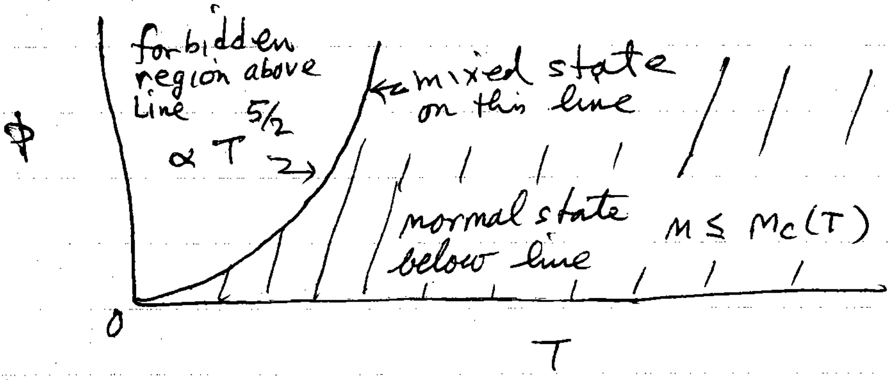
Define $n_c(T) = 2.612\left(\frac{2\pi m k_B T}{h^2}\right)^{3/2}$ inverse of $T_c(n)$

$n_c(T)$ is the critical density at a given T
— a system with $n > n_c(T)$ will be in a bose condensed mixed state at temperature T.

phase diagram in $p-T$ plane



Can also consider the transition in terms of p and $v = \frac{V}{N} = \frac{1}{n}$ for various fixed T.

At the transition $p \propto T_c(n)^{5/2}$, $T_c(n) \propto n^{2/3}$

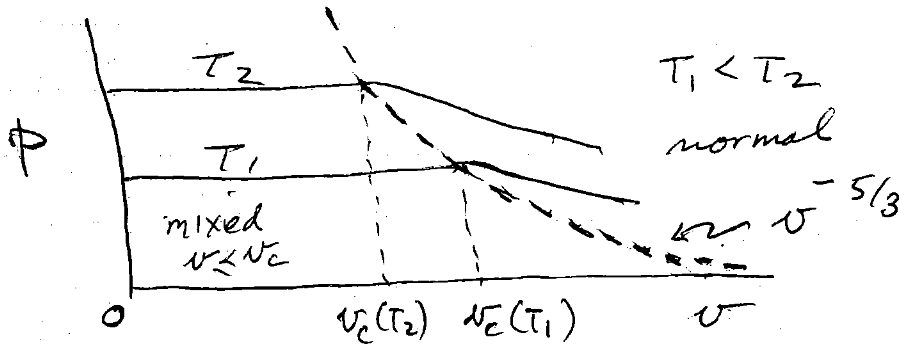
$\Rightarrow$ at the transition $p \propto (n^{2/3})^{5/2} = n^{5/3} = v^{-5/3}$

below the transition p is independent of density and hence independent of v.

For fixed T, the transition occurs when density n exceeds $n_c(T)$, or when v drops below $v_c(T) = \frac{1}{n_c(T)}$

$v_c(T) \sim T^{-3/2}$

curves of p vs v at constant T



Thermodynamic functions

Earlier we found $\frac{E}{V} = \frac{3}{2} p$

$$\Rightarrow \frac{E}{N} = \frac{3}{2} p \frac{V}{N} = \frac{3}{2} p v = \frac{3}{2} \frac{k_B T v}{\lambda^3} g_{5/2}(z)$$

$z = 1$ in mixed state
$z < 1$ in normal state

In above we regard $\frac{E}{N}$ as a function of either v or z. That is we either determine v for a given z, T or we determine z needed for a given v, T (Recall $z = e^{\beta\mu}$, $v = \frac{V}{N}$ and N and μ are conjugate variables)

specific heat

$$\frac{C_V}{N k_B} = \frac{1}{k_B}\left(\frac{\partial (E/N)}{\partial T}\right)_{V,N} = \frac{3}{2} v \left\{ \frac{d}{dT}\left(\frac{T}{\lambda^3}\right) g_{5/2}(z) + \frac{T}{\lambda^3} \frac{\partial g_{5/2}(z)}{\partial z} \frac{dz}{dT} \right.$$

For $T \le T_c$, $z = 1$ so $\frac{dz}{dT} = 0$ and only 1st term remains

$\frac{T}{\lambda^3} \propto T^{5/2}$ so $\frac{d}{dT}\left(\frac{T}{\lambda^3}\right) = \frac{5}{2}\left(\frac{T}{\lambda^3}\right)\frac{1}{T} = \frac{5}{2}\frac{1}{\lambda^3}$

$z=1$ here for all $T \le T_c$

$$\Rightarrow \frac{C_V}{Nk_B} = \frac{3}{2} v \left(\frac{5}{2}\frac{1}{\lambda^3}\right) g_{5/2}(1) = \frac{15}{4} g_{5/2}(1) \frac{v}{\lambda^3}$$

$$= \frac{15}{4} g_{5/2}(1)\, v \left(\frac{2\pi m k_B T}{h^2}\right)^{3/2}$$

Note, at T_c, $n = \frac{g_{3/2}(1)}{\lambda_c^3}$, and $v = \frac{1}{n}$

$$\frac{C_V(T_c^-)}{Nk_B} = \frac{15}{4}\frac{g_{5/2}(1)}{g_{3/2}(1)} = \frac{15}{4}\frac{1.341}{2.612} = 1.925$$

this is larger than the classical ideal gas value of $\frac{3}{2}$

So $$\boxed{\frac{C_V}{Nk_B} = 1.925 \left(\frac{T}{T_c}\right)^{3/2} \qquad T \le T_c}$$

For $T \ge T_c$, z varies with T and we need to evaluate the 2nd term as well

here z depends on T for $T > T_c$

1st term gives $\frac{15}{4} g_{5/2}(z(T)) \frac{v}{\lambda^3}$

2nd term: from Pathria Appendix D Eq(10),

$$z \frac{d}{dz}\left[g_\nu(z)\right] = g_{\nu-1}(z)$$

$$\Rightarrow \frac{d g_{5/2}}{dz}\frac{dz}{dT} = g_{3/2}\frac{1}{z}\frac{dz}{dT}$$

To find $\frac{1}{z}\frac{dz}{dT}$ consider our earlier result for the density when $T > T_c$:

$$n = \frac{g_{3/2}(z)}{\lambda^3} \quad \leftarrow \text{determines } z(T) \text{ for fixed } n$$

for n fixed $\Rightarrow$ $0 = \frac{dn}{dT} = \frac{d}{dT}\left(\frac{1}{\lambda^3}\right) g_{3/2} + \frac{1}{\lambda^3}\frac{dg_{3/2}}{dz}\frac{dz}{dT}$

$$0 = \frac{3}{2}\frac{1}{\lambda^3 T} g_{3/2} + \frac{1}{\lambda^3} g_{1/2} \frac{1}{z}\frac{dz}{dT}$$

$$\Rightarrow \frac{1}{z}\frac{dz}{dT} = -\frac{3}{2}\frac{g_{3/2}}{g_{1/2}}\frac{1}{T}$$

$$\frac{C_V}{Nk_B} = \frac{15}{4} g_{5/2}(z)\frac{v}{\lambda^3} + \frac{3}{2} v \frac{T}{\lambda^3} g_{3/2}(z)\left(\frac{-3}{2}\right)\frac{g_{3/2}(z)}{g_{1/2}(z)}\frac{1}{T}$$

use $n = \frac{1}{v} = \frac{g_{3/2}(z)}{\lambda^3} \Rightarrow \frac{v}{\lambda^3} = \frac{1}{g_{3/2}(z)}$

$$\boxed{\frac{C_V}{Nk_B} = \frac{15}{4}\frac{g_{5/2}(z)}{g_{3/2}(z)} - \frac{9}{4}\frac{g_{3/2}(z)}{g_{1/2}(z)} \qquad T > T_c}$$

Note $g_{1/2}(1) = \sum_{\ell=1}^{\infty} \frac{1}{\ell^{1/2}} \to \infty$

So as $T \to T_c^+$ from above, and $z \to 1$

$$\frac{C_V(T_c^+)}{Nk_B} = \frac{15}{4}\frac{g_{5/2}(1)}{g_{3/2}(1)} - \frac{9}{4}\frac{g_{3/2}(1)}{\infty} = \frac{15}{4}\frac{1.341}{2.612} = 1.925$$

$\Rightarrow$ $\boxed{C_V \text{ is continuous at } T_c}$

Finally we want to show that although C_V is continuous at T_c, $\frac{dC_V}{dT}$ is <u>discontinuous</u>

<u>For $T \le T_c$</u>

$$\frac{C_V}{Nk_B} = 1.925\left(\frac{T}{T_c}\right)^{3/2}$$

$$\frac{d}{dT}\left(\frac{C_V}{Nk_B}\right) = \frac{3}{2}(1.925)\left(\frac{T}{T_c}\right)^{1/2}\frac{1}{T_c} = 2.89\left(\frac{T}{T_c}\right)^{1/2}\frac{1}{T_c}$$

so slope at T_c^- (just below T_c) is:

$$\boxed{\frac{d}{dT}\left(\frac{C_V}{Nk_B}\right) = \frac{2.89}{T_c}, \quad T = T_c^-}$$

<u>For $T > T_c$</u>

$$\frac{C_V}{Nk_B} = \frac{15}{4}\frac{g_{5/2}(z)}{g_{3/2}(z)} - \frac{9}{4}\frac{g_{3/2}(z)}{g_{1/2}(z)}$$

$$\frac{d}{dT}\left(\frac{C_V}{Nk_B}\right) = \frac{15}{4}\,\frac{g_{3/2}\frac{dg_{5/2}}{dz}\frac{dz}{dT} - g_{5/2}\frac{dg_{3/2}}{dz}\frac{dz}{dT}}{(g_{3/2}(z))^2}$$

$$-\frac{9}{4}\,\frac{g_{1/2}\frac{dg_{3/2}}{dz}\frac{dz}{dT} - g_{3/2}\frac{dg_{1/2}}{dz}\frac{dT}{dT}}{(g_{1/2})^2}$$

$$= \frac{1}{z}\frac{dz}{dT}\left\{\frac{15}{4}\left(\frac{g_{3/2}^2 - g_{5/2}g_{1/2}}{g_{3/2}^2}\right) - \frac{9}{4}\left(\frac{g_{1/2}^2 - g_{3/2}g_{-1/2}}{g_{1/2}^2}\right)\right\}$$

use $\frac{1}{z}\frac{dz}{dT} = -\frac{3}{2}\frac{g_{3/2}}{g_{1/2}}\frac{1}{T}$ as found earlier

$$\frac{d}{dT}\left(\frac{C_V}{Nk_B}\right) = -\frac{3}{8T}\frac{g_{3/2}}{g_{1/2}}\left\{15\left(1-\frac{g_{5/2}\,g_{1/2}}{g_{3/2}^2}\right)-9\left(1-\frac{g_{3/2}\,g_{-1/2}}{g_{1/2}^2}\right)\right\}$$

Now as $T \to T_c^+$ from above, $z \to 1$, we have $g_{5/2}(1)$ and $g_{3/2}(1)$ are finite, but $g_{1/2}(1)$ and $g_{-1/2}(1) \to \infty$

$\Rightarrow$ at T_c^+

$$\frac{d}{dT}\left(\frac{C_V}{Nk_B}\right) = \frac{45}{8T_c}\frac{g_{5/2}(1)}{g_{3/2}(1)} - \frac{27}{8T_c}\,g_{3/2}^2(1)\,\frac{g_{-1/2}(1)}{g_{1/2}^3(1)}$$

Now from Pathria Appendix D Eq (8)

$$g_\nu(1) = \lim_{a\to 0}\frac{\Gamma(1-\nu)}{a^{1-\nu}}$$

So
$$\frac{g_{-1/2}(1)}{g_{1/2}^3(1)} = \lim_{a\to 0}\frac{\Gamma(3/2)}{a^{3/2}}\left(\frac{a^{1/2}}{\Gamma(1/2)}\right)^3 = \frac{\Gamma(3/2)}{[\Gamma(1/2)]^3}$$

$$= \frac{\frac{1}{2}\pi^{1/2}}{\pi^{3/2}} = \frac{1}{2\pi} \qquad \text{since } \Gamma(\tfrac{1}{2}) = \sqrt{\pi}, \quad \Gamma(\tfrac{3}{2}) = \tfrac{1}{2}\sqrt{\pi}$$

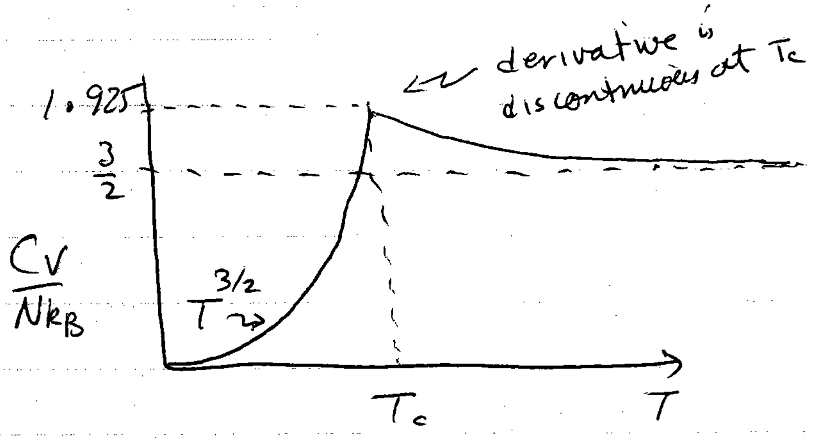
$$\frac{d}{dT}\left(\frac{C_V}{Nk_B}\right) = \frac{45}{8}\,\frac{1.341}{2.612}\,\frac{1}{T_c} - \frac{27}{8}\,\frac{(2.612)^2}{2\pi}\,\frac{1}{T_c}$$

$$= \frac{2.89}{T_c} - \frac{3.66}{T_c} = -\frac{0.77}{T_c}$$

$$\boxed{\frac{d}{dT}\left(\frac{C_V}{Nk_B}\right) = -\frac{0.77}{T_c}, \quad T = T_c^+}$$

slope of C_V is discontinuous at T_c.

C_V has a cusp at T_c



goes to classical $\frac{3}{2}$ as $T \to \infty$

$\frac{dC_V}{dT} > 0$ for $T = T_c^-$

$\frac{dC_V}{dT} < 0$ for $T = T_c^+$

Entropy

For single species gas we had for Gibbs free energy

$$G = N\mu$$

Also $G = E - TS + pV$ (since G is Legendre transform from E with respect to S and V)

$$\Rightarrow N\mu = E - TS + pV$$

or $$S = \frac{E + pV - N\mu}{T}$$

$$\frac{S}{Nk_B} = \frac{E + pV}{Nk_B T} - \frac{\mu}{k_B T}$$

we had earlier $E = \frac{3}{2} pV \Rightarrow pV = \frac{2}{3} E$

$$\frac{S}{Nk_B} = \frac{5}{3} \frac{E}{N} \frac{1}{k_B T} - \frac{\mu}{k_B T}$$

$$z = e^{\mu/k_BT} \quad , \quad z = 1 \quad \text{for } T < T_c$$

we had earlier $\frac{E}{N} = \frac{3}{2} \frac{k_BT\, v}{\lambda^3} g_{5/2}(z)$

and $n = \frac{1}{v} = \frac{g_{3/2}(z)}{\lambda^3}$ for $T > T_c$

$$\Rightarrow \frac{S}{Nk_B} = \frac{5}{2} \frac{v}{\lambda^3} g_{5/2}(z) - \ln z = \begin{cases} \frac{5}{2} \frac{g_{5/2}(z)}{g_{3/2}(z)} - \ln z & , T > T_c \\ \frac{5}{2} \frac{v}{\lambda^3} g_{5/2}(1) & , T \leq T_c \end{cases}$$

Note: For $T \leq T_c$ we had ~~that the density of the "normal"~~ a density $n_0 = n - \frac{g_{3/2}(1)}{\lambda^3}$ in the condensate, and a density $\frac{g_{3/2}(1)}{\lambda^3} \equiv n_n$ in the "normal" state (ie the density of excited particles)

$$\Rightarrow \text{for } T \leq T_c, \quad \frac{S}{Nk_B} = \frac{5}{2}\left(\frac{n_n}{n}\right)\frac{g_{5/2}(1)}{g_{3/2}(1)} \to 0 \quad \text{as } T \to 0$$

we can imagine that each normal particle carries entropy $\frac{5}{2} k_B \frac{g_{5/2}(1)}{g_{3/2}(1)}$. The entropy/per particle at $T < T_c$ is just the ~~fixed~~ above entropy per "normal" particle times the fraction of normal particles.

$\Rightarrow$ normal particles carry the entropy
condensate has zero entropy

entropy difference per particle between normal state and condensed state is $\frac{\Delta S}{N} = \frac{5}{2} k_B\, g_{5/2}(1)/g_{3/2}(1)$

latent heat of condensation

$$L = T\Delta S = \frac{5}{2} k_B T \frac{g_{5/2}(1)}{g_{3/2}(1)}$$

energy released upon converting one normal particle to one condensate particle.

⇒ mixed phase is like coexistence region of a 1st order phase transition (like water ⇔ ice – need to remove energy to turn water to ice)

⇒ "Two fluid" model of mixed region