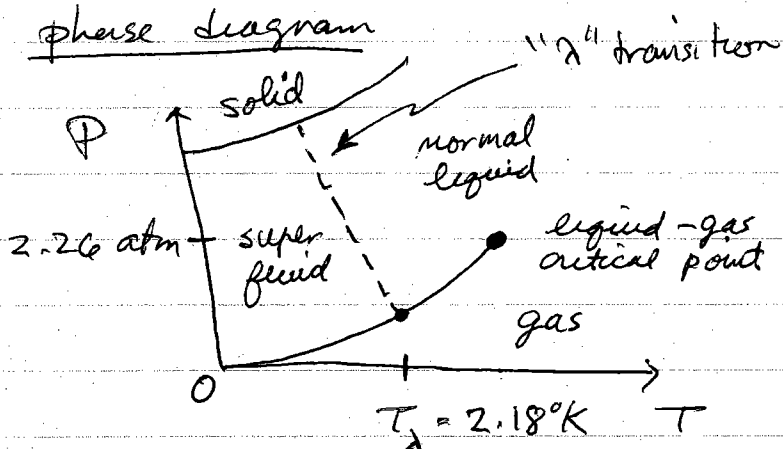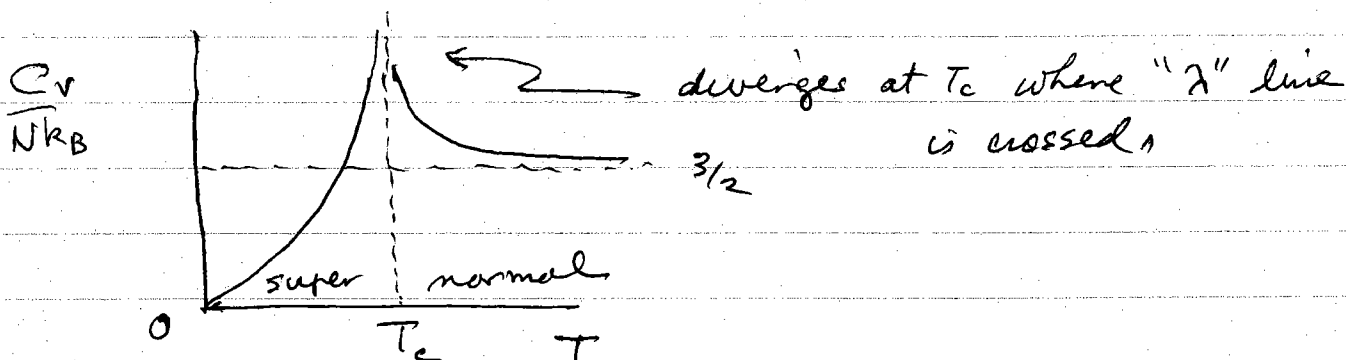# Super fluid ⁴He

phase diagram

liquid-gas transition
is 1st order — there is
a latent heat

normal-superfluid transition
is 2nd order — no latent
heat

specific heat

diverges at T_c where "λ" line
is crossed.

① Shape of C_v reminds one of the cusp in C_v
for Bose-Einstein condensation in ideal bose gas,
although here C_v diverges at T_c

② In superfluid phase, some fraction $\rho_s(T)/\rho$
[ρ_s = superfluid density, ρ = total density] of the fluid
flows without any dissipation and carries no entropy
— reminds one of the condensate of Bose-Einstein
Condensation.

③ For ^4He mass $m = 6.65 \times 10^{-24} g$, and ^4He fluid specific volume $v = 27.6 \; cm^3/mole$, one finds a Bose-Einstein condensation temperature of $T_c \simeq 3.13 \,^\circ K$ not far from $T_\lambda = 2.18 \,^\circ K$.

These observations suggested that super-fluidity in ^4He was related to Bose-Einstein condensation (BEC) of an ideal gas.

However

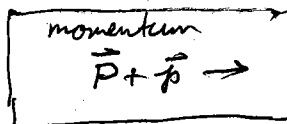① BEC transition line in P-T plane, $P_c \propto T^{5/2}$ has a positive slope, whereas λ-line of ^4He has a negative slope

② As demonstrated by <u>Landau</u> the condensate of BEC in an idea bose gas is <u>NOT</u> a superfluid.

Landau's arguement

Consider a condensate (all particles in same state) of total mass M, flowing with velocity $\vec{v}$ down a pipe. It has total momentum $\vec{P} = M\vec{v}$. In the rest frame of this fluid, consider an excitation that's created with momentum $\vec{p}$ and energy $\epsilon(\vec{p})$.
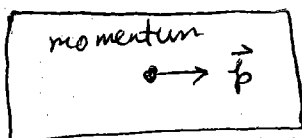
momentum $\longrightarrow \vec{p}$	momentum $\vec{P} + \vec{p} \rightarrow$
rest frame of fluid	rest frame of pipe

Back in the rest frame of the pipe, the change in energy due to the creation of this excitation is (in the limit of large total mass $M \rightarrow \infty$)

$$\Delta E = \frac{(\vec{P} + \vec{p})^2}{2M} + \epsilon(\vec{p}) - \frac{P^2}{2M}$$

$$= \frac{2 \vec{P} \cdot \vec{p}}{2M} + \frac{p^2}{2M} + \epsilon(\vec{p})$$

$$\Delta E = \epsilon(\vec{p}) + \vec{p} \cdot \vec{v} \qquad \text{since } \frac{\vec{P}}{M} = \vec{v}, \text{ and}$$

$$\frac{p^2}{2M} \rightarrow 0 \text{ for large } M$$

If $\Delta E < 0$, then it will be energetically favorable to create such excitations of momentum $\vec{p}$ — the motion of the fluid will therefore excite particles out of the condensate and degrade the flow, ie the system will _not_ display superfluidity.

$$\Delta E < 0 \implies \epsilon(\vec{p}) < -\vec{p} \cdot \vec{v}$$

$$\text{for } \vec{p} \parallel -\vec{v}, \implies \frac{\epsilon(p)}{p} < v$$

For an ideal bose gas, $\epsilon(p) = \frac{p^2}{2m}$, so

$$\frac{\epsilon(p)}{p} = \frac{p}{2m}$$ will always be less than flow velocity v, for sufficiently small p.

So the condensate of an ideal bose gas, flowing with any velocity v no matter how small, will always excite particles out of the condensate into low lying excited states, and hence will not be a superfluid.

However, from expt'l measurement such as neutron scattering and specific heat, it was known that the low lying excitation spectrum of ^4He was **not** free particle like, but rather as shown below
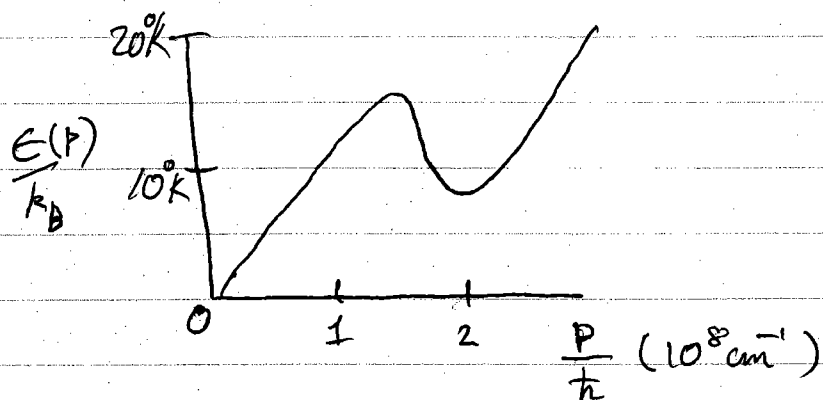
at small p, the spectrum is linear

$$\epsilon(p) \simeq cp$$

at higher p there is a dip

$$\epsilon(p) \simeq \Delta + \frac{(p-p_0)^2}{2m^*}$$

excitations in the low p linear region are phonons, or sound modes.
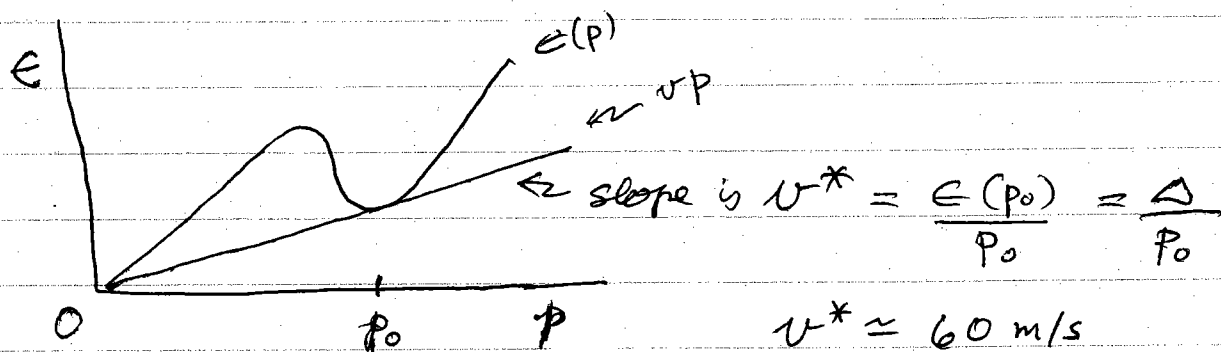excitations near the dip were called "rotons"

$$\frac{\Delta}{k_B} \simeq 8.7\,°K$$

$$c \simeq 2.4 \times 10^4 \, cm/s$$

$$m^* \simeq 0.16 \, m_{He}$$

$$p_{0_4} \simeq 1.9 \, Å^{-1}$$

As the fluid flow velocity v is increased from zero, the condition $\frac{\epsilon(p)}{p} = v$ will first be reached at the velocity v^* shown below

$\epsilon(p)$

$\leftarrow vp$

$\leftarrow$ slope is $v^* = \frac{\epsilon(p_0)}{p_0} = \frac{\Delta}{p_0}$

$v^* \simeq 60 \text{ m/s}$

for $v < v^*$ $\frac{\epsilon(p)}{p} > v$ for all values of p,

and so no particles will be excited out of the condensate by the flow. The fluid is now a superfluid!

Experimentally, while superfluidity does exist in 4He at low flow velocities, the critical velocity v_{crit}, above which the flow becomes dissipative, is much lower than the v^* above, and is strongly dependent on the size and shape of the container. This is due to the onset of turbulence accompanied by the proliferation of quantized vortices in the condensate flow.

The fact that the spectrum $\mathcal{E}(p)$ is linear at small p, and not free particle - like, is due to the fact that the ^4He atoms are NOT an ideal gas - in fact, the ^4He atoms are strongly interacting. This should not be surprising - the ^4He is a liquid and not a gas, particles collide with other giving rise to sound modes as the low lying excitations, rather than free "ballistic" motion of individual atoms.

One can show that for an interacting bose gas, even for weak interactions, the small p part of the excitation spectrum always is phonon - like, ie linear, (see Pathria §10.3 or ~~take~~ PHY509!). Superfluidity can only exist for interacting bose systems.

Because interactions between ^4He atoms are not weak, there is no good microscopic theory of superfluidity in ^4He. Calculations can, however, be performed for weak interactions. This has lead people to look for superfluidity or BEC in weakly interacting bose gases. This search was finally successful with the observation of BEC in magnetically trapped, laser cooled, dilute atomic gases (produced two Nobel prizes!)

Bose-Einstein Condensation in laser cooled gases

gases of alkali atoms Li, Na, K, Rb, Cs
- all have a single s-electron in outermost shell.
  — important for efficiency of laser cooling
- use isotopes such that total intrinsic spin of all electrons and nucleons add up to an integer F_i
  $\Rightarrow$ atoms are $\underline{bosons}$
- all have a net magnetic moment — used to confine dilute gas of atoms in a "magnetic trap"
- use "laser cooling" to get $\underline{very}$ low temperatures in low density gases, to try and see BEC

magnetic trap $\rightarrow$ effective harmonic potential for atoms

$$V(r) = \tfrac{1}{2} m \omega_0^2 r^2 \qquad \omega_0 \cong 2\pi \times 100 \, Hz$$

1995 — 10^3 atoms with $T_c \sim 100 \, nK$

1999 — 10^8 atoms with $T_c \sim \mu K$ gas size $\sim$ many microns

How was BEC in these systems observed?

energy levels of ideal (non-interacting) bosons in harmonic trap

$$e(n_x, n_y, n_z) = (n_x + n_y + n_z + \tfrac{3}{2}) \hbar \omega_0$$
$$n_x, n_y, n_z \text{ integers}$$

ground state condensate wavefunction
$$\psi_0(r) \sim e^{-r^2/2a^2} \quad \text{with} \quad a = \left(\frac{\hbar}{m\omega_0}\right)^{1/2}$$
$a \sim 1 \, \mu m$ for current traps

$\Rightarrow$ condensate has spatial extent $\sim a$

The spatial extent of the nth excited energy level
is roughly

$$m\omega_0^2 \langle r^2 \rangle \sim \epsilon(n) \propto n\hbar\omega_0$$

$$\Rightarrow \langle r^2 \rangle \sim \frac{n\hbar}{m\omega_0} \qquad \text{or} \qquad \sqrt{\langle r^2 \rangle} = \left(\frac{n\hbar}{m\omega_0}\right)^{1/2}$$

For $k_B T \gg \hbar\omega_0$, the atoms are excited
up to level $n \sim \frac{k_B T}{\hbar\omega_0}$

$\Rightarrow$ spatial extent of the normal component of the
gas is

$$R \sim \left(\frac{n\hbar}{m\omega_0}\right)^{1/2} \sim \left(\frac{\hbar k_B T}{\hbar m\omega_0^2}\right)^{1/2} = \left(\frac{k_B T}{m\omega_0}\right)^{1/2}$$

$$R \sim a \left(\frac{k_B T}{\hbar\omega_0}\right)^{1/2} \gg a$$

If T_c is the BEC transition temperature, then for
$T > T_c$ one sees a more or less uniform cloud
of atoms with radius $R \sim a (k_B T/\hbar\omega_0)^{1/2} \gg a$.
But when one cools to $T < T_c$, one now has a
finite fraction of the atoms condensed in the ground
state. $\rightarrow$ superimposed on the atomic cloud of radius
R one sees the growth of a sharp peak in density
at the center of cloud — this peak has a radius $a \ll R$.

To find T_c, $\qquad\qquad$ (use $z=1$ at $T \leq T_c$)

for $T \leq T_c$

$$m = M_0 + \int_0^\infty dn_x \int_0^\infty dn_y \int_0^\infty dn_z \frac{1}{e^{(n_x+n_y+n_z)\hbar\omega_0/k_B T}-1}$$

$$= M_0 + \left(\frac{k_B T}{\hbar\omega_0}\right)^3 \int_0^\infty dx \int_0^\infty dy \int_0^\infty dz \frac{1}{e^{(x+y+z)}-1}$$

$$= M_0 + \left(\frac{k_B T}{\hbar\omega_0}\right)^3 \zeta(3)$$

at T_c, $M_0 = 0 \Rightarrow k_B T_c = \hbar\omega_0 \left(\frac{m}{\zeta(3)}\right)^{1/3}$

condensate density $\qquad M_0(T) = m\left(1 - \left(\frac{T}{T_c}\right)^3\right)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow$

$\qquad\qquad\qquad\qquad\qquad\qquad$ different from ideal free

$\qquad\qquad\qquad\qquad\qquad\qquad$ gas due to presence of

$\qquad\qquad\qquad\qquad\qquad\qquad$ magnetic trapping potential

$$\zeta(3) = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots$$