Relation between canonical and microcanonical A

We now investigate the effect that the energy fluctuations have on the canonical Helmholtz free energy A, as compared to the microcanonical Helmholtz free energy

microcanonical A:

① compute $S(E) = k_B \ln \Omega(E)$ from the microcanonical partition function $\Omega(E)$

② take Legendre transform of S with respect to E to get $-\frac{A}{T} = S - \frac{E}{T}$ this gives the microcanonical A

We will write the Legendre transform as follows:

$$-\frac{A(T)}{T} = \max_E \left[S(E) - \frac{E}{T} \right]$$

or $$A(T) = \min_E \left[E - TS(E) \right]$$

let $\bar{E}$ be this minimizing value of E

$$A_{micro} = \bar{E} - TS(\bar{E})$$

Canonical A

① compute $A(T) = -k_B T \ln Q_N(T)$

Consider now the computation of $Q_N = e^{-A/k_B T}$

$$Q_N = e^{-A/k_BT} = \int \frac{dE}{\Delta}\, \Omega(E)\, e^{-E/k_BT} \qquad \text{use } S = k_B \ln \Omega$$

$$= \int \frac{dE}{\Delta}\, e^{S(E)/k_B}\, e^{-E/k_BT}$$

$$= \int \frac{dE}{\Delta}\, e^{-(E - TS(E))/k_BT}$$

Consider the exponent $E - TS(E)$ and expand to 2^{nd} order about its minimum at $\bar{E}$. $E = \bar{E} + \delta E$

$$E - TS(E) = \underbrace{\bar{E} - TS(\bar{E})}_{0^{th}\text{ order}} + \underbrace{\delta E - T\left(\frac{\partial S}{\partial E}\right)_{V,N} \delta E}_{1^{st}\text{ order}} \underbrace{- \frac{1}{2} T \left(\frac{\partial^2 S}{\partial E^2}\right)_{V,N} \delta E^2}_{2^{nd}\text{ order}}$$

$$= A_{micro} + \underbrace{\delta E - T\left(\frac{1}{T}\right)\delta E}_{\text{cancel}} - \frac{1}{2} T \left(\frac{\partial (1/T)}{\partial E}\right)_{V,N} \delta E^2$$

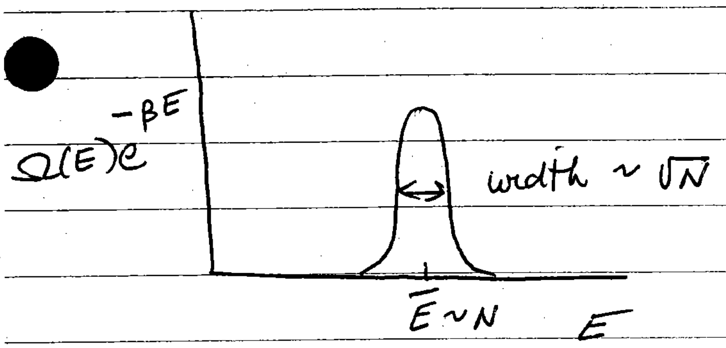
$$= A_{micro} + \frac{1}{2}\frac{1}{T}\left(\frac{\partial T}{\partial E}\right)_{V,N} \delta E^2$$

$$= A_{micro} + \frac{\delta E^2}{2 T C_V} \qquad \text{where used } \left(\frac{\partial T}{\partial E}\right)_{V,N} = \frac{1}{\left(\frac{\partial E}{\partial T}\right)_{V,N}} = \frac{1}{C_V}$$

$$Q_N = e^{-A/k_BT} = \int \frac{d\,\delta E}{\Delta}\, e^{-A_{micro}/k_BT}\, e^{-\delta E^2/2k_BT^2C_V}$$

we have a Gaussian integral – integrand is sharply peaked at $\delta E = 0$ with width $\sqrt{\langle \delta E^2 \rangle} = \sqrt{k_B T^2 C_V} \sim \sqrt{N}$

so $\dfrac{\sqrt{\langle \delta E^2 \rangle}}{\bar{E}} \sim \dfrac{\sqrt{N}}{N} = \dfrac{1}{\sqrt{N}}$ small fluctuations



We can do the Gaussian integration to explicitly evaluate Q_N

use $\int dx\, e^{-x^2/2\sigma^2} = \sqrt{2\pi\sigma^2}$

$$Q_N = e^{-A/k_BT} = e^{-A_{micro}/k_BT} \frac{\sqrt{2\pi k_B T^2 C_V}}{\Delta}$$

take logs

$$A = A_{micro} - k_BT \ln\left(\frac{\sqrt{2\pi k_B T^2 C_V}}{\Delta}\right)$$

$$A = A_{micro} - \frac{1}{2}k_BT \ln\left(\frac{2\pi k_B T^2 C_V}{\Delta^2}\right)$$

canonical Helmholtz free energy — microcanonical Helmholtz free energy — correction due to fluctuations in energy

Note: $A \sim A_{micro} \sim N$, $C_V \sim N$

so the correction term between A and A_{micro} has relative size

$$\frac{A - A_{micro}}{A} \simeq \frac{\ln N}{N} \to 0 \quad \text{as } N \to \infty$$

$\Rightarrow$ the canonical ensemble gives the same results as the microcanonical ensemble, provided one takes the <u>thermodynamic limit</u> $N \to \infty$.

this is because as $N \to \infty$, the most probable energy $\bar{E}$ is the same as the average energy $\langle E \rangle$, and all other energies have negligible propability to occur.

Average energy $\langle E \rangle$ vs. the most probable energy $\bar{E}$ in the canonical ensemble.

In our earlier discussion of fluctuations in the canonical ensemble we expanded

$$E - TS(E) \cong A_{micro}(T) + \frac{\delta E^2}{2TC_V}$$

now we continue the expansion to $O(\delta E^3)$

$$E - TS(E) \cong A_{micro}(T) + \frac{\delta E^2}{2TC_V} - \frac{1}{3!} T \frac{\partial^3 S}{\partial E^3}\bigg|_{E=\bar{E}} \delta E^3$$

Note $\frac{\partial^3 S}{\partial E^3} \sim \frac{1}{N^2}$ since $S \sim N$ and $E \sim N$ are both extensive

so we can write

$$E - TS(E) \cong A_{micro}(T) + \frac{\delta E^2}{2TC_V} - \frac{\gamma}{N^2} \delta E^3$$

where γ is some constant that does not increase with N (it can depend on T)

Now compute $\langle \delta E \rangle = \langle E \rangle - \bar{E}$

$$\langle \delta E \rangle = \frac{\int \frac{d\delta E}{\Delta} e^{-(E - TS(E))/k_B T} \delta E}{\int \frac{d\delta E}{\Delta} e^{-(E - TS(E)/k_B T)}}$$

$$\simeq \frac{\int d\delta E \; e^{-\frac{\delta E^2}{2k_B T^2 C_V} + \frac{\gamma \delta E^3}{N^2 k_B T}} \delta E}{\int d\delta E \; e^{-\frac{\delta E^2}{2k_B T^2 C_V} + \frac{\gamma}{N^2}\frac{\delta E^3}{k_B T}}}$$

$$\simeq \frac{\int d\delta E \; e^{-\frac{\delta E^2}{2k_B T^2 C_V}} \left(1 + \frac{\gamma \delta E^3}{N^2 k_B T}\right) \delta E}{\int d\delta E \; e^{-\frac{\delta E^2}{2k_B T^2 C_V}} \left(1 + \frac{\gamma \delta E^3}{N^2 k_B T}\right)}$$

where we expanded $e^{\frac{\gamma \delta E^3}{N^2 k_B T}} \approx 1 + \frac{\gamma \delta E^3}{N^2 k_B T}$ for $N \to \infty$

For a Gaussian distribution, only the even moments are non-vanishing

$$\langle \delta E \rangle \simeq \frac{\int d\delta E \; e^{-\frac{\delta E^2}{2k_B T^2 C_V}} \cdot \left(\frac{\gamma}{N^2 k_B T}\right) \delta E^4}{\int d\delta E \; e^{-\frac{\delta E^2}{2k_B T^2 C_V}}}$$

$$= \left(\frac{\gamma}{N^2 k_B T}\right) \cdot \left(k_B T^2 C_V\right)^2 \cdot 3$$

where we used
$$\frac{\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}\, x^4}{\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}} = 3\sigma^4$$

But the main point is $C_V \sim N$

so $$\langle \delta E \rangle \sim \frac{1}{N^2} \cdot N^2 \sim O(1)$$

The relative difference between average and most probable energy therefore scales as

$$\frac{\langle E \rangle - \bar{E}}{\langle E \rangle} = \frac{\langle \delta E \rangle}{\langle E \rangle} \sim \frac{1}{N} \to 0 \quad \text{as } N \to \infty$$

Stirling's Formula

In lecture we used the saddle point approx to discuss the relation between the Helmholtz free energy in the canonical vs. the micro canonical ensemble. The saddle pt approx is also how one derives Stirling's approx for $n!$

Consider the integral

$$I = \int_0^\infty dx\, x^n e^{-x}$$

integrate by parts

$$I = -x^n e^{-x}\Big]_0^\infty + \int_0^\infty n x^{n-1} e^{-x} dx$$

boundary term vanishes at its limits so

$$I = \int_0^\infty dx\, n x^{n-1} e^{-x}$$

integrate by parts again

$$I = \int_0^\infty dx\, n(n-1) x^{n-2} e^{-x}$$

and so on to get

$$I = \int_0^\infty dx\, n(n-1)(n-2)\cdots(1)\, e^{-x} = n!$$

Now evaluate I in saddle pt approx.

Define $U(x) = -x + n \ln x$

$$I = \int_0^\infty dx \; e^{U(x)}$$

expand $U(x)$ about its maximum

$$U'(x) = -1 + \frac{n}{x} \quad \Rightarrow \begin{cases} U(\bar{x}) = -n + n \ln n \\ \bar{x} = n \quad \text{is the maximum} \end{cases}$$

$$U''(x) = -\frac{n}{x^2} \quad \Rightarrow \quad U''(\bar{x}) = -\frac{1}{n}$$

$$U'''(x) = \frac{2n}{x^3} \qquad U'''(\bar{x}) = 2/n^2$$

$$U''''(x) = \frac{-6n}{x^4} \qquad U''''(\bar{x}) = -6/n^3$$

For $\delta x = x - \bar{x}$,

$$U(x) \simeq -n + n \ln n - \frac{\delta x^2}{2n} + \frac{1}{6}\frac{2\,\delta x^3}{n^2} - \frac{1}{24}\frac{6\,\delta x^4}{n^3} + \ldots$$

$$= -n + n \ln n - \frac{\delta x^2}{2n} + \frac{\delta x^3}{3n^2} - \frac{\delta x^4}{4n^3} + \ldots$$

$$I = \int_0^\infty dx \; e^{-n + n \ln n} \, e^{-\delta x^2/2n} \, \underbrace{e^{\frac{\delta x^3}{3n^2} - \frac{\delta x^4}{4n^3}}}_{\text{expand for small } \delta x}$$

$$\approx \int_{-\infty}^{\infty} d\delta x \; e^{-n+n\ln n} \, e^{-\delta x^2/2n} \left[1 + \frac{\delta x^3}{3n^2} - \frac{\delta x^4}{4n^3} + O(\delta x^6) \right]$$

$$= e^{-n + n \ln n} \int_{-\infty}^{\infty} d\delta x \; e^{-\delta x^2/2n} \left[1 + \frac{\delta x^3}{3n^2} - \frac{\delta x^4}{4n^3} + \ldots \right]$$

$$= e^{-n+n\ln n} \sqrt{2\pi n} \left[1 + \frac{\langle \delta x^3 \rangle}{3n^2} - \frac{\langle \delta x^4 \rangle}{4n^3} + \ldots \right]$$

Now $\langle \delta x^3 \rangle = 0$, $\langle \delta x^4 \rangle \sim n^2$, So

$$I = n! = e^{-n + n\ln n} \sqrt{2\pi n} \left[1 + O\left(\tfrac{1}{n}\right) \right]$$

$$\begin{aligned} \ln n! &= n\ln n - n + \tfrac{1}{2}\ln n + \tfrac{1}{2}\ln 2\pi + \ln\left(1 + O\left(\tfrac{1}{n}\right)\right) \\ &= \underbrace{n\ln n - n}_{\text{these are the leading terms}} + \underbrace{\tfrac{1}{2}\ln n + \tfrac{1}{2}\ln 2\pi}_{\text{these are next order corrections}} + O\left(\tfrac{1}{n}\right) \end{aligned}$$

Factorization of canonical partition function
– the ideal gas

Consider a system of N noninteracting particles

$$\Rightarrow \mathcal{H}[\vec{q}_i, \vec{p}_i] = \sum_{i=1}^{N} \mathcal{H}^{(1)}(\vec{q}_i, \vec{p}_i)$$

where $\mathcal{H}^{(1)}$ is the single particle Hamiltonian that depends only on the three coordinates $\vec{q}_i$ and three momenta $\vec{p}_i$ of particle i.

$$Q_N = \frac{1}{N!\, h^{3N}} \left(\prod_{i=1}^{N} \int d\vec{q}_i \, d\vec{p}_i \right) e^{-\beta \mathcal{H}}$$

$$= \frac{1}{N!} \left(\prod_{i=1}^{N} \int \frac{d\vec{q}_i \, d\vec{p}_i}{h^3} \right) e^{-\beta \sum_j \mathcal{H}^{(1)}(\vec{q}_j, \vec{p}_j)}$$

factor the exponential

$$= \frac{1}{N!} \prod_{i=1}^{N} \left(\int \frac{d\vec{q}_i \, d\vec{p}_i}{h^3} e^{-\beta \mathcal{H}^{(1)}(\vec{q}_i, \vec{p}_i)} \right)$$

factor for particle i is identical to factor for particle j

$$\Rightarrow \boxed{Q_N = \frac{1}{N!} (Q_1)^N}$$ for noninteracting particles

where Q_1 is the one particle partition function

$$Q_1 = \int \frac{d\vec{q}\,d\vec{p}}{h^3}\, e^{-\beta \mathcal{H}^{(1)}(\vec{q},\vec{p})}$$

Apply to the ideal gas.

$$\mathcal{H}^{(1)}(\vec{q},\vec{p}) = \frac{p^2}{2m}$$

$$Q_1 = \int \frac{d\vec{q}}{h^3} \int d\vec{p}\; e^{-\beta \frac{p^2}{2m}}$$

$$\int d\vec{q} = V \quad \text{volume of system}$$

$$\int d\vec{p}\; e^{-\beta p^2/2m} = \left(\frac{2\pi m}{\beta}\right)^{3/2} \quad \text{3D Gaussian integral}$$

$$Q_1 = \frac{V}{h^3}\,(2\pi m k_B T)^{3/2}$$

$$\Rightarrow Q_N = \frac{1}{N!}\left(\frac{V}{h^3}\right)^N (2\pi m k_B T)^{3N/2}$$

$$\ln N! = N \ln N - N$$

$$A(T,V,N) = -k_B T \ln Q_N \quad \text{using } \underbrace{\text{Stirling's}}\text{ formular}$$

$$= -k_B T \left\{ N \ln\left[\frac{V}{h^3}(2\pi m k_B T)^{3/2}\right] - N \ln N + N \right\}$$

$$A(T,V,N) = -k_B T N - k_B T N \ln\left[\frac{V}{h^3 N}(2\pi m k_B T)^{3/2}\right]$$

Compute average energy

$$\langle E\rangle = -\frac{\partial}{\partial\beta}(\ln Q_N) = -\frac{\partial}{\partial\beta}(-\beta A)$$

$$= -\frac{\partial}{\partial\beta}\left(N + N\ln\left[\frac{V}{h^3N}\left(\frac{2\pi m}{\beta}\right)^{3/2}\right]\right)$$

$$= -N\frac{\partial}{\partial\beta}\left(\ln\beta^{-3/2}\right) = \frac{3}{2}N\frac{\partial}{\partial\beta}\ln\beta = \frac{3}{2}N\frac{1}{\beta}$$

$\langle E\rangle = \frac{3}{2}Nk_BT$ as expected

entropy

$$S = -\left(\frac{\partial A}{\partial T}\right)_{V,N} = k_BN + k_BN\ln\left[\frac{V}{h^3N}(2\pi mk_BT)^{3/2}\right]$$
$$+ k_BTN\frac{3}{2}\left(\frac{1}{T}\right)$$

from derivative of log

$$S = \frac{5}{2}Nk_B + Nk_B\ln\left[\frac{V}{h^3N}(2\pi mk_BT)^{3/2}\right]$$

substitute in $k_BT = \frac{2}{3}\frac{E}{N}$ to get

$$\Rightarrow \quad S(E,V,N) = \frac{5}{2}Nk_B + Nk_B\ln\left[\frac{V}{h^3N}\left(\frac{4\pi mE}{3N}\right)^{3/2}\right]$$

we have recovered the Sackur-Tetrode equation which we earlier derived from the microcanonical ensemble! Canonical and microcanonical approaches are equivalent.

Because in computing Q_N we sum over all states with any energy, as compared to computing Ω where we restrict the sum to states in a particular energy shell E, it is usually easier to compute Q_N, rather than Ω.

We introduced the canonical distribution as a means of describing a physical system in contact with a heat bath.

The canonical distrib gives the same result as the microcanonical because in the $N \to \infty$ (thermodynamic) limit, the canonical probability distribution

$$p(E) = \frac{\Omega(E)\, e^{-E/k_B T}}{\Delta\, Q_N(V,T)}$$

approaches a delta-function* at the most probable energy = average energy, as set by the temperature T.

We could alternatively introduced the canonical ensemble just as a mathematical trick for computing $\Omega(E)$, removing the constraint of constant energy E by means of a Lagrange multiplier.

* since $E \sim N$ increases as $N \to \infty$

and $\langle E^2\rangle - \langle E\rangle^2$ increases as $\sqrt{N}$

it is not really $P(E)$ that approaches a well defined function as $N \to \infty$. Rather it is the distribution

$p(e \equiv E/N)$, the probability density to have an <u>energy per particle</u> e, that approaches a delta function as $N \to \infty$

Note:

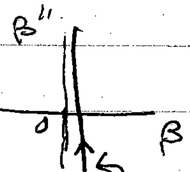
$$Q_N(\beta) = \int \frac{dE}{\Delta}\, \Omega(E)\, e^{-\beta E}$$

$Q_N(\beta)$ is Laplace transform of $\frac{\Omega}{\Delta}(E)$

$\Rightarrow \frac{\Omega}{\Delta}$ is inverse Laplace transform of Q_N

$$\frac{\Omega(E)}{\Delta} = \frac{1}{2\pi i}\int_{\beta'-i\infty}^{\beta'+i\infty} e^{\beta E}\, Q_N(\beta)\, d\beta \qquad (\beta' > 0)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\beta'+i\beta'')}\, Q_N(\beta'+i\beta'')\, d\beta''$$



where $\beta' = \mathrm{Re}(\beta) = 0^+$

contour of integration lies to right of imaginary axis

entropy $S = k_B \ln \Omega$

Helmholtz $-\frac{A}{T} = k_B \ln Q_N$

$$-\frac{A}{T} = S - \frac{E}{T}$$

Helmholtz free energy is Legendre transf of S with respect to E

Thermodynamic potentials which are Legendre transforms of each other, have ensemble partition functions that are Laplace transforms of each other.

Virial Theorem — Classical Systems Only

Consider $\langle x_i \frac{\partial \mathcal{H}}{\partial x_j} \rangle = \dfrac{\int dq_i dp_i \, x_i \frac{\partial \mathcal{H}}{\partial x_j} e^{-\beta \mathcal{H}}}{\int dq_i dp_i \, e^{-\beta \mathcal{H}}}$

where x_i and x_j are any of the 6N generalized coordinates q, p $i = 1, \ldots, 3N$.

$$\int dq_i \, dp_i \; x_i \frac{\partial \mathcal{H}}{\partial x_j} e^{-\beta \mathcal{H}} = -\frac{1}{\beta} \int dq_i \, dp_i \; x_i \frac{\partial}{\partial x_j} \left(e^{-\beta \mathcal{H}} \right)$$

integrate by parts with ~~r~~ respect to x_j

$$= -\frac{1}{\beta} \int' dq_i \, dp_i \; x_i e^{-\beta \mathcal{H}} \Big]_{x_j^{(1)}}^{x_j^{(2)}} + \frac{1}{\beta} \int dq_i \, dp_i \left(\frac{\partial x_i}{\partial x_j} \right) e^{-\beta \mathcal{H}}$$

integral over all coordinates *except* x_j

$x_j^{(1)}$ and $x_j^{(2)}$ are the extremal values of x_j

the boundary integral vanishes because $\mathcal{H}$ becomes infinite at the extremal values of any coordinate

- if x_j is a momentum p, then extremal values are $p = \pm\infty$ and $\mathcal{H} \propto p^2/2m \to \infty$.
- if x_j is a spatial coord q, then extremal values are at boundary of ~~sten~~ system, where the potential energy confining the particle to the volume V becomes infinite.

$$\Rightarrow \int dq_i \, dp_i \; x_i \frac{\partial \mathcal{H}}{\partial x_j} e^{-\beta \mathcal{H}} = \frac{1}{\beta} \int dq_i \, dp_i \left(\frac{\partial x_i}{\partial x_j} \right) e^{-\beta \mathcal{H}}$$

but $\frac{\partial x_i}{\partial x_j} = \delta_{ij}$

$$\Rightarrow \quad \langle x_i \frac{\partial \mathcal{H}}{\partial x_j} \rangle = \frac{1}{\beta} \delta_{ij} \frac{\int dq_i \int dp_i \, e^{-\beta \mathcal{H}}}{\int dq_i \int dp_i \, e^{-\beta \mathcal{H}}}$$

$$\boxed{\langle x_i \frac{\partial \mathcal{H}}{\partial x_j} \rangle = k_B T \, \delta_{ij}} \Leftarrow \text{Virial Theorem}$$

If $x_i = x_j = p_i$ then

$$\langle p_i \frac{\partial \mathcal{H}}{\partial p_i} \rangle = \langle p_i \dot{q}_i \rangle = k_B T$$

If $x_i = x_j = q_i$, then

$$\langle q_i \frac{\partial \mathcal{H}}{\partial q_i} \rangle = -\langle q_i \dot{p}_i \rangle = k_B T$$

where we used Hamiltons equ's of motion
$\partial \mathcal{H}/\partial p_i = \dot{q}_i$ and $\partial \mathcal{H}/\partial q_i = -\dot{p}_i$

$$\Rightarrow \quad \langle \sum_{i=1}^{3N} p_i \dot{q}_i \rangle = 3N k_B T$$

$$-\langle \sum_{i=1}^{3N} q_i \dot{p}_i \rangle = 3N k_B T$$ — Virial Theorem Clausius (1870)

Equipartition theorem — Classical systems only

Suppose the Hamiltonian is quadratic in some particular degree of freedom x_j (x_j is either a coord or a momentum)

$$\mathcal{H}[q_i, p_i] = \mathcal{H}'[q_i, p_i] + \alpha_j x_j^2$$

depends on all degrees of freedom <u>except</u> x_j

Then $\langle \mathcal{H} \rangle = \langle \mathcal{H}' \rangle + \alpha_j \langle x_j^2 \rangle$

contribution to total average energy from the degree of freedom x_j

$$\langle x_j^2 \rangle = \frac{\prod_i \int dq_i\, dp_i\; x_j^2\, e^{-\beta(\mathcal{H}' + \alpha_j x_j^2)}}{\prod_i \int dq_i\, dp_i\; e^{-\beta(\mathcal{H}' + \alpha_j x_j^2)}}$$

$$= \frac{\left(\prod_i{}' \int dq_i\, dp_i\, e^{-\beta \mathcal{H}'}\right)}{\left(\prod_i{}' \int dq_i\, dp_i\, e^{-\beta \mathcal{H}'}\right)} \; \frac{\int dx_j\; x_j^2\, e^{-\beta \alpha_j x_j^2}}{\int dx_j\; e^{-\beta \alpha_j x_j^2}}$$

where $\prod_i{}'$ is over all degrees of freedom <u>except</u> x_j

$$\langle x_j^2 \rangle = \frac{\int dx_j \, x_j^2 e^{-\beta \alpha_j x_j^2}}{\int dx_j \, e^{-\beta \alpha_j x_j^2}} = \frac{1}{2\beta\alpha_j} = \frac{1}{2}\frac{k_B T}{\alpha_j}$$

(follows from $\int dx \, e^{-x^2/2\sigma^2} = \sqrt{2\pi\sigma^2}$ and $\dfrac{\int dx \, e^{-x^2/2\sigma^2} x^2}{\sqrt{2\pi\sigma^2}} = \sigma^2$)

So the contribution to $\langle H \rangle$ from the degree of freedom x_j

is $\alpha_j \langle x_j^2 \rangle = \alpha_j \frac{1}{2} \frac{k_B T}{\alpha_j} = \frac{1}{2} k_B T$

$\Rightarrow$ each quadratic degree of freedom in the Hamiltonian contributes $\frac{1}{2} k_B T$ to the total average energy.

Ideal gas: $\quad H = \sum_{i=1}^{N} \frac{|\vec{p}_i|^2}{2m}$

There are $3N$ quadratic degrees of freedom: the three momenta $\vec{p}_i$ components for each particle

$$\Rightarrow E = \langle H \rangle = \frac{3N}{2} k_B T$$

or average energy per particle

$$\langle \epsilon \rangle = \frac{E}{N} = \frac{3}{2} k_B T$$

as we saw earlier from the simple kinetic theory of the ideal gas