The previous examples of phonons in a solid and Black Body radiation were problems involving <u>bosons</u> with excitation spectrum

$\epsilon = \hbar\omega = \hbar c|\vec{k}|$ (ie linear spectrum)

and zero chemical potential $\mu = 0$.

non-interacting

Now we want to turn to the problem of an ideal quantum gas (bosons <u>or</u> fermions) of physical particles with ~~an excitation~~ an ordinary non-relativistic excitation spectrum

$$\epsilon = \frac{\hbar^2 k^2}{2m} \quad \text{(ie quadratic spectrum)}$$

and $\mu \neq 0$.

Ideal Quantum Gas – Grand canonical ensemble

$$\ln \mathcal{Z} = \pm \sum_i \ln\left(1 \pm e^{-\beta(\epsilon_i - \mu)}\right) \qquad +\text{FD},\ -\text{BE}$$

for free particles, states can be labeled by ~~wavevector~~ wavevector $\vec{k}$ with $k_\mu = \frac{2\pi n_\mu}{L}$, $n_\mu = 0, \pm 1, \pm 2, \ldots$ due to periodic boundary conditions. volume $V = L^3$

$$\Rightarrow \underset{\text{states}}{\sum_i} \rightarrow \sum_s \sum_{\vec{k}} \rightarrow g_s \frac{V}{(2\pi)^3} \int_0^\infty dk\, 4\pi k^2$$

spin polarizations

\# spin states for each $\vec{k}$

for free particles, ϵ depends only on $|\vec{k}|$. Define density of states $g(\epsilon)$ such that

$$\frac{g_s}{(2\pi)^3} \int dk\, 4\pi k^2 = \int g(\epsilon)\, d\epsilon$$

$g(\epsilon)$ = # states with energy ϵ per unit energy per volume

$$\Rightarrow g(\epsilon) = \frac{g_s 4\pi}{(2\pi)^3} k^2 \frac{dk}{d\epsilon}$$

For non-relativistic particles $\epsilon = \frac{\hbar^2 k^2}{2m}$, $k = \sqrt{\frac{2m\epsilon}{\hbar^2}}$

$$g(\epsilon) = \frac{g_s 4\pi}{(2\pi)^3} \frac{2m\epsilon}{\hbar^2} \sqrt{\frac{2m}{\hbar^2}} \frac{1}{2\sqrt{\epsilon}}$$

$$= \frac{2\pi g_s}{(2\pi)^3} \left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{\epsilon} = \left(\frac{2\pi m}{h^2}\right)^{3/2} \frac{2^{3/2} (2\pi)^{3/2}}{(2\pi)^2} g_s \sqrt{\epsilon}$$

Density of States

$$\boxed{g(\epsilon) = \left(\frac{2\pi m}{h^2}\right)^{3/2} \frac{2 g_s}{\sqrt{\pi}} \sqrt{\epsilon}} \qquad g \sim \sqrt{\epsilon}$$

pressure

$$\frac{p}{k_B T} = \frac{1}{V}\ln \mathcal{L} = \pm \frac{1}{V}\sum_i \ln\left(1 \pm z\, e^{-\beta \epsilon_i}\right) \qquad z = e^{\beta\mu}$$

$$= \pm \int_0^\infty d\epsilon\; g(\epsilon)\, \ln\left(1 \pm z e^{-\beta\epsilon}\right)$$

$$= \pm \left(\frac{2\pi m}{h^2}\right)^{3/2} \frac{2 g_s}{\sqrt{\pi}} \int_0^\infty d\epsilon\; \sqrt{\epsilon}\, \ln\left(1 \pm z e^{-\beta\epsilon}\right)$$

substitute variables $y = \beta E$

$$\frac{p}{k_B T} = \pm \left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} \frac{2 g_s}{\sqrt{\pi}} \int_0^\infty dy\; y^{1/2} \ln\left(1 \pm z e^{-y}\right)$$

integrate by parts $\qquad \lambda = \left(\frac{h^2}{2\pi m k_B T}\right)^{1/2}$ thermal wavelength

$$\frac{p}{k_B T} = \pm \frac{2 g_s}{\sqrt{\pi}\,\lambda^3} \left\{ \underbrace{\frac{2}{3} y^{3/2} \ln\left(1 \pm z e^{-y}\right)\Big|_0^\infty}_{=0} - \int_0^\infty dy\, \frac{2}{3} y^{3/2} \frac{(\mp z e^{-y})}{1 \pm z e^{-y}} \right\}$$

$$\boxed{\frac{p}{k_B T} = \frac{4 g_s}{3\sqrt{\pi}\,\lambda^3} \int_0^\infty dy\, \frac{y^{3/2}}{z^{-1} e^{y} \pm 1}} \qquad \begin{array}{l} + \text{FD} \\ - \text{BE} \end{array}$$

density of particles $\quad \frac{N}{V} = \sum_i \langle n_i \rangle$

$$\frac{N}{V} = \frac{1}{V}\sum_i \frac{1}{z^{-1} e^{\beta\epsilon_i} \pm 1} = \int_0^\infty d\epsilon\; g(\epsilon) \frac{1}{z^{-1} e^{\beta\epsilon} \pm 1}$$

$$= \left(\frac{2\pi m}{h^2}\right)^{3/2} \frac{2 g_s}{\sqrt{\pi}} \int_0^\infty d\epsilon\, \frac{\sqrt{\epsilon}}{z^{-1} e^{\beta\epsilon} \pm 1}$$

$$= \left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} \frac{2 g_s}{\sqrt{\pi}} \int dy\, \frac{y^{1/2}}{z^{-1} e^{y} \pm 1}$$

$$\boxed{\frac{N}{V} = \frac{2 g_s}{\sqrt{\pi}\,\lambda^3} \int_0^\infty dy\, \frac{y^{1/2}}{z^{-1} e^{y} \pm 1}} \qquad \begin{array}{l} + \text{FD} \\ - \text{BE} \end{array}$$

energy density $\quad E = \sum_i \epsilon_i \langle n_i \rangle$

$$\frac{E}{V} = \frac{1}{V}\sum_i \frac{\epsilon_i}{z^{-1}e^{\beta\epsilon_i} \pm 1} = \int_0^\infty d\epsilon\, g(\epsilon)\frac{\epsilon}{z^{-1}e^{\beta\epsilon}\pm 1}$$

$$= \frac{2g_s}{\sqrt{\pi}\lambda^3} k_BT \int_0^\infty dy \frac{y^{3/2}}{z^{-1}e^{y}\pm 1}$$

$$\frac{E}{V} = \frac{3}{2}k_BT \frac{4g_s}{3\sqrt{\pi}\lambda^3}\int_0^\infty \frac{y^{3/2}}{z^{-1}e^{y}\pm 1} = \left(\frac{3}{2}k_BT\right)\left(\frac{p}{k_BT}\right)$$

$$\Rightarrow \frac{E}{V} = \frac{3}{2}p \quad \text{or} \quad \boxed{p = \frac{2}{3}\frac{E}{V}}$$

both fermions and bosons (same result as for classical ideal gas!!)

Define "standard functions" (see Pathria Appendices D and E)

$$f_n(z) \equiv \frac{1}{\Gamma(n)}\int_0^\infty dy \frac{y^{n-1}}{z^{-1}e^{y}+1} = \sum_{\ell=1}^{\infty}(-1)^{\ell+1}\frac{z^\ell}{\ell^n}$$

$$g_n(z) = \frac{1}{\Gamma(n)}\int_0^\infty dy \frac{y^{n-1}}{z^{-1}e^{y}-1} = \sum_{\ell=1}^{\infty}\frac{z^\ell}{\ell^n}$$

$\Gamma(n+1) = n\,\Gamma(n)$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$\Rightarrow \Gamma(3/2) = \frac{1}{2}\sqrt{\pi}$

$\Gamma(5/2) = \frac{3}{4}\sqrt{\pi}$

In terms of these:

Fermions	Bosons
$\frac{p}{k_BT} = \frac{g_s}{\lambda^3} f_{5/2}(z)$	$\frac{p}{k_BT} = \frac{g_s}{\lambda^3} g_{5/2}(z)$
$\frac{N}{V} = \frac{g_s}{\lambda^3} f_{3/2}(z)$	$\frac{N}{V} = \frac{g_s}{\lambda^3} g_{3/2}(z)$
$\frac{E}{V} = \frac{3}{2}k_BT\frac{g_s}{\lambda^3} f_{5/2}(z)$	$\frac{E}{V} = \frac{3}{2}k_BT\frac{g_s}{\lambda^3} g_{5/2}(z)$
$\frac{E}{N} = \frac{3}{2}k_BT \frac{f_{5/2}(z)}{\ldots}$	$\frac{E}{N} = \frac{3}{2}k_BT\frac{g_{5/2}(z)}{g_{3/2}(z)}$

Equation of state: low densities - virial expansion

$z \ll 1$ "non-degenerate"

keep lowest terms in series expansion

$$\frac{p}{k_B T} = \frac{g_s}{\lambda^3}\left\{\begin{matrix} f_{5/2} \\ g_{5/2}\end{matrix}\right\} = \frac{g_s}{\lambda^3}\left(z \mp \frac{z^2}{2^{5/2}} + \cdots\right) \quad \begin{matrix} - \text{ FD} \\ + \text{ BE}\end{matrix}$$

$$\frac{N}{V} = \frac{g_s}{\lambda^3}\left\{\begin{matrix} f_{3/2} \\ g_{3/2}\end{matrix}\right\} = \frac{g_s}{\lambda^3}\left(z \mp \frac{z^2}{2^{3/2}} + \cdots\right)$$

$$\Rightarrow \frac{p}{k_B T} = \frac{N}{V}\frac{\left(z \mp \frac{z^2}{2^{5/2}} + \cdots\right)}{\left(z \mp \frac{z^2}{2^{3/2}} + \cdots\right)} = \frac{N}{V}\left(1 \mp \frac{z}{2^{5/2}} + \cdots\right)\left(1 \pm \frac{z}{2^{3/2}} + \cdots\right)$$

$$= \frac{N}{V}\left(1 \pm \frac{z}{2^{3/2}} \mp \frac{z}{2^{5/2}} + \cdots\right) \qquad \frac{1}{2^{3/2}} - \frac{1}{2^{5/2}} = \frac{2}{2^{5/2}} - \frac{1}{2^{5/2}} = \frac{1}{2^{5/2}}$$

$$pV = Nk_B T\left(1 \pm \frac{z}{2^{5/2}} + \cdots\right)$$

quantum correction to classical ideal gas law.

+ FD - p increases compared to classically
- effective repulsion due to Pauli exclusion

− BE - p decreases compared to classically
- effective attraction.

above is similar conclusion to what we saw from 2-particle density matrix.

for small z, the leading term gives $\frac{N}{V} = \frac{g_s}{\lambda^3} z$

or $z = \left(\frac{N}{V}\frac{\lambda^3}{g_s}\right)$ - same result we had classically

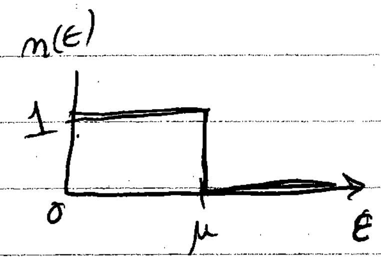
$\Rightarrow$ small z limit is the low density limit $n\lambda^3 \ll 1$ $\equiv$ or high T

$$pV = Nk_B T\left(1 \pm \frac{1}{2^{5/2} g_s}\frac{N}{V}\lambda^3 + \cdots\right)$$

Sommerfeld model of electrons in a conductor

Fermi gas – high density / low temperature limit
"degenerate" fermi gas

Consider first $T \to 0$



$$\langle n(\epsilon)\rangle = \frac{1}{e^{\beta(\epsilon-\mu)}+1}$$

as $T \to 0$ $\quad e^{\beta(\epsilon-\mu)} \to \begin{cases} \infty & \epsilon > \mu \\ 0 & \epsilon < \mu \end{cases}$

$$\Rightarrow \langle n(\epsilon)\rangle \to \begin{cases} 0 & \epsilon > \mu \\ 1 & \epsilon < \mu \end{cases}$$

$\Rightarrow$ all states with $\epsilon < \mu$ are filled, all states with $\epsilon > \mu$ are empty. This is the $T=0$ ground state of the Fermi gas. We therefore see that $\mu(T=0)$ is the energy of the highest energy single particle state that is occupied in the ground state. One calls this energy the Fermi-energy

$$\epsilon_F \equiv \mu(T=0)$$

at $T=0$

$$N = g_s \sum_{\vec{k} \,\leftarrow\, \text{s.t. } \frac{\hbar^2 k^2}{2m} \leq \epsilon_F} 1 \qquad \text{count occupied states}$$

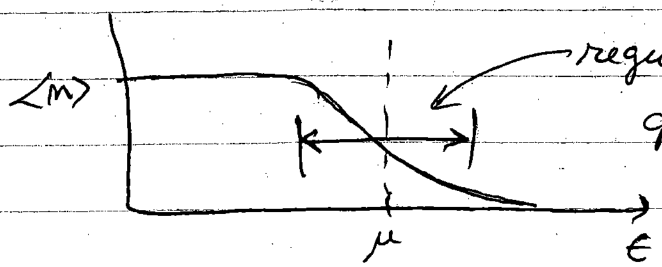
$$= g_s \frac{V}{(2\pi)^3} 4\pi \int_0^{k_F} dk\, k^2 = \frac{g_s V}{6\pi^2} k_F^3 \qquad \text{where } \frac{\hbar^2 k_F^2}{2m} = \epsilon_F$$

$$n \equiv \frac{N}{V} = \frac{g_s}{6\pi^2} k_F^3 = \frac{g_s}{6\pi^2}\left(\frac{2m\epsilon_F}{\hbar^2}\right)^{3/2}$$

or $\quad \epsilon_F = \frac{\hbar^2}{2m}\left(\frac{6\pi^2 n}{g_s}\right)^{2/3}, \quad k_F = \left(\frac{6\pi^2 n}{g_s}\right)^{1/3}$

relation between $\mu(T=0)$ and density $n = N/V$

Now at finite T



region of energy where $\langle n \rangle$ differs from ground state ($T=0$) is a region of order $k_B T$ about μ.

So the $T \simeq 0$ approx is good when $k_B T \ll \mu$

~~Since $\mu(T) > \mu(0) = \epsilon_F$ we have~~

Using $\mu \approx \mu(0) = \epsilon_F$ we have

$$k_B T \ll \frac{\hbar^2}{2m}\left(\frac{6\pi^2 n}{g_s}\right)^{2/3} \Rightarrow \frac{2\pi m k_B T}{h^2} \ll \frac{1}{4\pi}\left(\frac{6\pi^2 n}{g_s}\right)^{2/3}$$

$$\Rightarrow \lambda^2 \gg 4\pi\left(\frac{g_s}{6\pi^2 n}\right)^{2/3}$$

$$\Rightarrow n\lambda^3 \gg \frac{(4\pi)^{3/2}}{6\pi^2} g_s = \frac{4}{3\sqrt{\pi}} g_s$$

so this is equivalent to a low T <u>or</u> a high density limit $n\lambda^3 \gg 1$ – called the "degenerate" limit.
(just as the classical limit $z \simeq n\lambda^3 \ll 1$ was a high T low density limit)

Fermi temperature $T_F = \epsilon_F / k_B$. Degenerate limit is $T \ll T_F$

For electrons in a metal, $T_F \simeq 10000\,°K$.
So electrons in a metal are always in the degenerate limit.