

## Energy in the degenerate limit $T=0$

$$\frac{E}{V} = \int_0^{\epsilon_F} d\epsilon g(\epsilon) \epsilon$$

$$g(\epsilon) = C \sqrt{\epsilon}$$

$$\text{with } C = \left(\frac{2\pi m}{h^2}\right)^{3/2} \frac{2g_s}{\sqrt{\pi}}$$

$$n = \frac{N}{V} = \int_0^{\epsilon_F} d\epsilon g(\epsilon)$$

↖ density of states

$$\Rightarrow \frac{E}{V} = C \int_0^{\epsilon_F} d\epsilon \epsilon^{3/2} = \frac{2}{5} C \epsilon_F^{5/2}$$

$$n = \frac{N}{V} = C \int_0^{\epsilon_F} d\epsilon \epsilon^{1/2} = \frac{2}{3} C \epsilon_F^{3/2}$$

$$\Rightarrow \frac{E}{V} = \frac{3}{5} \frac{N}{V} \epsilon_F$$

$$\frac{E}{V} = \frac{3}{5} n \epsilon_F$$

or

$$\boxed{\frac{E}{N} = \frac{3}{5} \epsilon_F}$$

↑ energy per volume

↑ energy per particle

Above gives  $T=0$  results. To get behavior at low  $T > 0$ , or to get quantities such as  $C_V = \left(\frac{\partial E}{\partial T}\right)_V$ , we need to get the next order terms in a low temperature expansion.

In general we need to do integrals of the form

$$\int d\epsilon \frac{\tilde{\phi}(\epsilon)}{z^{-1} e^{\beta\epsilon} + 1} = \int d\epsilon \tilde{\phi}(\epsilon) n(\epsilon), \quad \tilde{\phi}(\epsilon) \text{ some function}$$

ex: to compute  $n$ ,  $\tilde{\phi}(\epsilon) = g(\epsilon)$ ; to compute  $\frac{E}{V}$ ,  $\tilde{\phi}(\epsilon) = g(\epsilon) \epsilon$

transform variables to  $X = \beta \epsilon$ .

then we want to do integrals of the form

$$\Phi \equiv \int_0^{\infty} dx \frac{\phi(x)}{z^{-1} e^x + 1}$$

$\phi(x)$  is any function of  $x$ .

For example, to get the "standard" function  $f_n(z)$ , we use  $\phi(x) = \frac{1}{\Gamma(n)} x^{n-1}$

Define  $\xi = \beta \mu = \ln z$

$$\Phi = \int_0^{\infty} dx \frac{\phi(x)}{e^{x-\xi} + 1}$$

Define  $\psi(x) \equiv \int_0^x \phi(x') dx'$ ,  $f(x) \equiv \frac{1}{[e^{x-\xi} + 1]}$  Fermi function

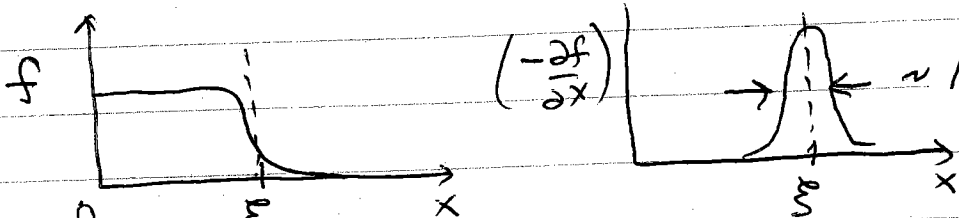
$$\Phi = \int_0^{\infty} dx \left( \frac{\partial \psi}{\partial x} \right) f(x) \quad \text{integrate by parts}$$

$$= \psi(x) f(x) \Big|_0^{\infty} + \int_0^{\infty} dx \psi(x) \left( -\frac{\partial f}{\partial x} \right)$$

$$= \int_0^{\infty} dx \psi(x) \left( -\frac{\partial f}{\partial x} \right) \quad \text{since } \psi(0) = 0 \text{ and } f(\infty) = 0$$

1st term vanishes

Now we use the fact that at low  $T$ ,  $\left( -\frac{\partial f}{\partial x} \right)$  is strongly peaked about  $x = \xi$



$\xi \gg 1$   
 $\xi \sim \frac{E_F}{kT}$  large

expand  $\psi(x)$  about  $x = \xi$

$$\psi(x) = \sum_{n=0}^{\infty} \left. \frac{d^n \psi}{dx^n} \right|_{x=\xi} \frac{(x-\xi)^n}{n!}$$

$$\Rightarrow \Phi = \sum_{n=0}^{\infty} \left. \frac{d^n \psi}{dx^n} \right|_{x=\xi} \int_0^{\infty} dx \frac{(x-\xi)^n}{n!} \left( \frac{-\partial f}{\partial x} \right)$$

since  $\left( \frac{-\partial f}{\partial x} \right)$  is zero except for a region of order 1

about  $x = \xi \gg 1$ , we can replace the lower limit of the integral by  $-\infty$  without any noticeable change

Then we can make a change of variables  $y = x - \xi$   
and the integrals become

$$\int_{-\infty}^{\infty} dy \frac{y^n}{n!} \left( \frac{-\partial f}{\partial y} \right) \quad \text{where } f(y) = \frac{1}{e^y + 1}$$

$$\text{Now } \frac{-\partial f}{\partial y} = \frac{e^y}{(e^y + 1)^2} = \frac{e^y}{e^{2y} + 2e^y + 1} = \frac{1}{e^y + 2 + e^{-y}}$$

is symmetric about  $y = 0$ ,

$\Rightarrow$  all the integrals for  $n$  odd vanish!

To sum over only n even terms, let  $n \rightarrow 2n$

$$\Phi = \sum_{n=0}^{\infty} \frac{d^{2n} \phi}{dx^{2n}} \Big|_{x=\xi} \int_{-\infty}^{\infty} dy \frac{y^{2n}}{(2n)!} \left( -\frac{\partial f}{\partial y} \right)$$

$$\text{let } a_n \equiv \int_{-\infty}^{\infty} dy \frac{y^{2n}}{(2n)!} \left( -\frac{\partial f}{\partial y} \right), \quad a_0 = \int_{-\infty}^{\infty} dy \left( -\frac{\partial f}{\partial y} \right) = 1$$

The  $a_n$  are just numbers that be computed.  
They contain no system parameters whatsoever

For  $n \geq 1$  one can show

$$a_n = 2 \left( 1 - \frac{1}{2^{2n}} + \frac{1}{3^{2n}} - \frac{1}{4^{2n}} + \frac{1}{5^{2n}} - \dots \right)$$

$$= \left( 2 - \frac{1}{2^{2(n-1)}} \right) \zeta(2n)$$

where  $\zeta(n) = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \dots$  is the Riemann zeta function

$$\text{In particular } a_1 = \frac{\pi^2}{6}, \quad a_2 = \frac{7\pi^4}{360}$$

$$\Phi = \sum_{n=0}^{\infty} a_n \frac{d^{2n} \phi}{dx^{2n}} \Big|_{x=\xi} = \phi(\xi) + \sum_{n=1}^{\infty} a_n \frac{d^{2n} \phi}{dx^{2n}} \Big|_{x=\xi}$$

$$\text{use } \frac{d\phi}{dx} = \phi \text{ to finally get } \quad \phi(x) = \int_0^x dx' \phi(x')$$

$$\Phi = \int_0^{\xi} dx \phi(x) + \sum_{n=1}^{\infty} a_n \frac{d^{2n-1} \phi}{dx^{2n-1}} \Big|_{x=\xi}$$

$$= \int_0^{\xi} dx \phi(x) + \frac{\pi^2}{6} \frac{d\phi}{dx} \Big|_{x=\xi} + \frac{7\pi^4}{360} \frac{d^3 \phi}{dx^3} \Big|_{x=\xi} + \dots$$

This gives a power series in temperature.

To see this, transform back to the energy variable

$$x = \beta \epsilon, \quad \epsilon = k_B T x$$

$$\Phi \equiv \int_0^{\infty} d\epsilon \frac{\phi(\epsilon)}{z^{-1} e^{\beta \epsilon} + 1} = k_B T \left\{ \int_0^{\infty} dx \frac{\phi(k_B T x)}{z^{-1} e^x + 1} \right\}$$

Using  $\xi = \mu/k_B T$

$$k_B T \int_0^{\mu} dx \phi(k_B T x) = \int_0^{\mu} d\epsilon \phi(\epsilon)$$

and  $\frac{d\phi}{dx} = \frac{d\phi}{d\epsilon} \frac{d\epsilon}{dx} = \frac{d\phi}{d\epsilon} k_B T$

We get

$$\Phi = \int_0^{\infty} d\epsilon \phi(\epsilon) m(\epsilon)$$

$$\Phi = \int_0^{\mu} d\epsilon \phi(\epsilon) + \frac{\pi^2}{6} (k_B T)^2 \left. \frac{d\phi}{d\epsilon} \right|_{\epsilon=\mu} + \frac{7\pi^4}{360} (k_B T)^4 \left. \frac{d^3 \phi}{d\epsilon^3} \right|_{\epsilon=\mu} + \dots$$

Example

① density  $n = \frac{N}{V} = \int_0^{\infty} d\epsilon g(\epsilon) m(\epsilon) \Rightarrow \phi(\epsilon) \equiv g(\epsilon)$

$$n = \int_0^{\mu} d\epsilon g(\epsilon) + \frac{\pi^2}{6} (k_B T)^2 \left. \frac{dg}{d\epsilon} \right|_{\epsilon=\mu} + \dots$$

Now as  $T \rightarrow 0$ ,  $\mu \rightarrow \epsilon_F$  the fermi energy

$$n = \int_0^{\epsilon_F} d\epsilon g(\epsilon) + \int_{\epsilon_F}^{\mu} d\epsilon g(\epsilon) + \frac{\pi^2}{6} (k_B T)^2 \left. \frac{dg}{d\epsilon} \right|_{\epsilon=\mu}$$

But  $\epsilon_F$  was determined by  $n = \int_0^{\epsilon_F} d\epsilon g(\epsilon)$

$$\Rightarrow \int_{\epsilon_F}^{\mu} d\epsilon g(\epsilon) = -\frac{\pi^2}{6} (k_B T)^2 \left. \frac{dg}{d\epsilon} \right|_{\epsilon=\mu}$$

since left hand side is  $o(k_B T)^2$  is small, we can approximate ~~the right hand side~~ as it as

$$\int_{\epsilon_F}^{\mu} d\epsilon g(\epsilon) \approx (\mu - \epsilon_F) g(\epsilon_F)$$

$$\Rightarrow (\mu - \epsilon_F) \approx -\frac{\pi^2}{6} \frac{(k_B T)^2}{g(\epsilon_F)} \left. \frac{dg}{d\epsilon} \right|_{\epsilon=\mu}$$

so  $\mu - \epsilon_F \sim o(k_B T)^2$  is small, so to lowest order can evaluate  $\frac{dg}{d\epsilon}$  on right hand side at  $\epsilon = \epsilon_F$  instead of  $\epsilon = \mu$

$$\boxed{\mu(T) \approx \epsilon_F - \frac{\pi^2}{6} (k_B T)^2 \frac{g'(\epsilon_F)}{g(\epsilon_F)}} \quad g' = \frac{dg}{d\epsilon}$$

Shows that chemical potential  $\mu$  decreases from  $\epsilon_F$  by  $o(k_B T)^2$  at low  $T$

For free electrons where  $g(\epsilon) = C\sqrt{\epsilon}$   
 $g'(\epsilon) = \frac{1}{2}C\frac{1}{\sqrt{\epsilon}}$

$$\mu(T) \approx \epsilon_F - \frac{\pi^2}{6} (k_B T)^2 \frac{1}{2\epsilon_F} = \epsilon_F - \frac{\pi^2}{12} \frac{(k_B T)^2}{\epsilon_F}$$

$$\mu(T) \approx \epsilon_F \left( 1 - \frac{1}{3} \left( \frac{\pi k_B T}{2\epsilon_F} \right)^2 \right) = \epsilon_F \left( 1 - \frac{1}{3} \left( \frac{\pi T}{2 T_F} \right)^2 \right)$$

Correction is small for metals at room temp as  $T_F \sim 10,000^\circ\text{K}$

② energy  $\frac{E}{V} = \int_0^\infty d\epsilon g(\epsilon) \epsilon n(\epsilon) \Rightarrow \phi(\epsilon) = g(\epsilon) \epsilon$

$$u = \frac{E}{V} = \int_0^\mu d\epsilon g(\epsilon) \epsilon + \frac{\pi^2}{6} (k_B T)^2 [g(\mu) + \mu g'(\mu)]$$

$$= \underbrace{\int_0^{\epsilon_F} d\epsilon g(\epsilon) \epsilon}_{= u(0)} + \underbrace{\int_{\epsilon_F}^\mu d\epsilon g(\epsilon) \epsilon}_{\approx (\mu - \epsilon_F) g(\epsilon_F) \epsilon_F} + \frac{\pi^2}{6} (k_B T)^2 [g(\mu) + \mu g'(\mu)]$$

ground state energy density
as before
replace  $\mu \approx \epsilon_F$  as before

$$u(T) = u(0) + (\mu - \epsilon_F) g(\epsilon_F) \epsilon_F + \frac{\pi^2}{6} (k_B T)^2 [g(\epsilon_F) + \epsilon_F g'(\epsilon_F)]$$

$$= u(0) + \left[ \frac{-\pi^2}{6} (k_B T)^2 \frac{g'(\epsilon_F)}{g(\epsilon_F)} \right] g(\epsilon_F) \epsilon_F + \frac{\pi^2}{6} (k_B T)^2 [g(\epsilon_F) + \epsilon_F g'(\epsilon_F)]$$

$$u(T) = u(0) + \frac{\pi^2}{6} (k_B T)^2 g(\epsilon_F)$$

specific heat per volume

$$c_v \equiv \frac{C_v}{V} = \frac{1}{V} \left( \frac{dE}{dT} \right) = \left( \frac{dU}{dT} \right) / V$$

$$c_v = \frac{\pi^2}{3} k_B^2 T g(E_F)$$

for free electrons we can write  $g(E) = C\sqrt{E}$

$$n = \int_0^{E_F} dE g(E) = \frac{2}{3} C E_F^{3/2} \Rightarrow C = \frac{3}{2} \frac{n}{E_F^{3/2}}$$

$$\Rightarrow g(E_F) = \frac{3}{2} \frac{n}{E_F^{3/2}} \cdot E_F^{1/2} = \frac{3}{2} \frac{n}{E_F} \quad \text{density of states at fermi energy}$$

$$c_v = \frac{\pi^2}{2} \left( \frac{k_B T}{E_F} \right) n k_B$$

or total specific heat  $C_v = V c_v$   $nV = N$

$$C_v = \frac{\pi^2}{2} \left( \frac{k_B T}{E_F} \right) N k_B$$

$\Rightarrow$  specific heat due to fermi gas of electrons in a conductor is  $C_v \sim T$  at low temperatures

We already saw that specific heat due to ionic vibrations (phonons) in a solid went like  $C_v \sim T^3$  at low temperatures (Debye model)

$\Rightarrow$  electronic contribution to  $C_v$  dominates at sufficiently low  $T$ .



electronic specific heat per volume

$$C_V^{\text{elec}} = \frac{\pi^2}{2} \left( \frac{k_B T}{E_F} \right) \frac{N k_B}{V} \left( 1 + o \left( \frac{k_B T}{E_F} \right)^2 \right)$$

compare to classical result  $C_V^{\text{classical}} = \frac{N k_B}{V}$

The correct result for degenerate fermi gas is a factor

$$\frac{\pi^2}{2} \left( \frac{k_B T}{E_F} \right) = \frac{\pi^2}{2} \left( \frac{T}{T_F} \right) \text{ smaller than classical result by factor } \sim \frac{10^2}{10^4} = 10^{-2} \text{ at room temperature}$$

also, classical  $C_V$  is indep of  $T$ , whereas fermi gas result is  $\propto T$ .

At low  $T$ , the ionic contribution to  $C_V$  is

$$C_V^{\text{ion}} = \frac{12\pi^4}{5} \left( \frac{T}{\Theta_D} \right)^3 \frac{N k_B}{V}$$

$$\frac{C_V^{\text{elec}}}{C_V^{\text{ion}}} = \frac{\pi^2}{2} \left( \frac{k_B T}{E_F} \right) \frac{5}{12\pi^4} \left( \frac{\Theta_D}{T} \right)^3 \approx \frac{5}{24\pi^2} \left( \frac{\Theta_D}{T_F} \right) \left( \frac{\Theta_D}{T} \right)^2$$

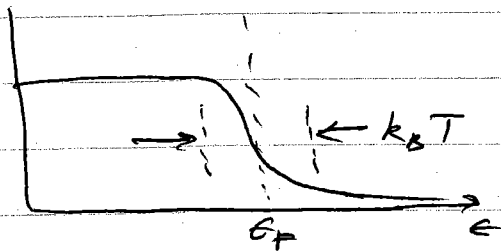
$$\approx 1 \text{ when } T^* = \sqrt{\frac{5}{24\pi^2} \left( \frac{\Theta_D}{T_F} \right)} \Theta_D \approx 0.15 \left( \frac{\Theta_D}{T_F} \right)^{1/2} \Theta_D$$

for metals,  $T_F \sim 10^4$  °K,  $\Theta_D \sim 10^2$  °K

$$T^* = 0.15 \sqrt{10^{-2}} \Theta_D \approx 0.015 \Theta_D$$

So ionic contrib to  $C_V$  dominates over electronic contrib until  $T \lesssim 0.01 \Theta_D$  i.e. at 0(1) °K. The electronic contrib dominates at lower temperatures.

## Simple estimate of $C_V$



When increase temperature to  $k_B T$ , the electrons near the Fermi energy  $\epsilon_F$  will increase their energy by an amount  $\sim k_B T$ . The number of such electrons ~~is roughly~~ per unit volume is roughly

$$g(\epsilon_F)(k_B T)$$

↑ density of states at  $\epsilon_F$       ↑ energy interval about  $\epsilon_F$  of states which ~~increase~~ get excited

⇒ increase in energy per unit volume is

$$\Delta U \sim \left( \underset{\substack{\uparrow \\ \# \text{ electrons} \\ \text{excited}}}{g(\epsilon_F) k_B T} \right) \left( \underset{\substack{\uparrow \\ \text{excitation} \\ \text{energy per} \\ \text{excited electron}}}{k_B T} \right) \sim g(\epsilon_F) (k_B T)^2$$

$$\Rightarrow C_V = \frac{d\Delta U}{dT} \sim g(\epsilon_F) k_B^2 T$$

The previous calculation gives the precise numerical coefficient

## Pauli paramagnetism of electron gas

$$\vec{S} = \frac{1}{2} \hbar \vec{\sigma}$$

electron has intrinsic spin  $\vec{S}$  with intrinsic magnetic moment  $\vec{\mu} = -\mu_B \vec{\sigma}$   $\mu_B = \frac{e\hbar}{2mc}$  is Bohr magneton

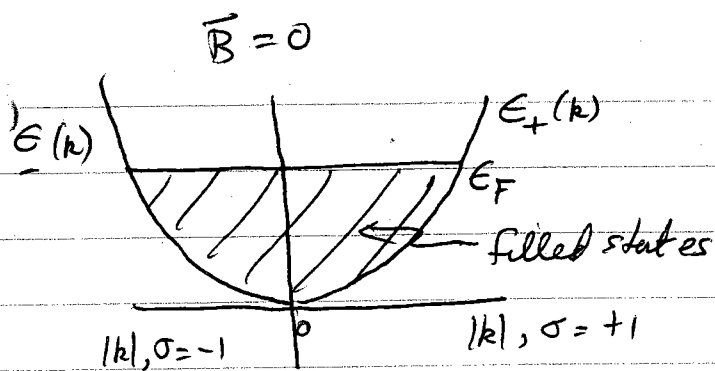
In an external magnetic field  $\vec{B}$ , there is an interaction energy  $-\vec{\mu} \cdot \vec{B} = \mu_B \sigma B$  where  $\sigma = \pm 1$  for spins parallel and antiparallel to  $\vec{B}$ . The energy spectra for up and down electron spins becomes

$$E_{\pm}(\vec{k}) = E(\vec{k}) \pm \mu_B B \quad \text{where } E(\vec{k}) \text{ is spectrum at } \vec{B} = 0$$

Since  $\uparrow$  and  $\downarrow$  electrons now have different energy spectra, we should treat them as two different populations of particles  $\Rightarrow$  they will be in equilibrium when their chemical potentials are equal, i.e.  $\mu_+ = \mu_-$

this will induce a net magnetization in the system.

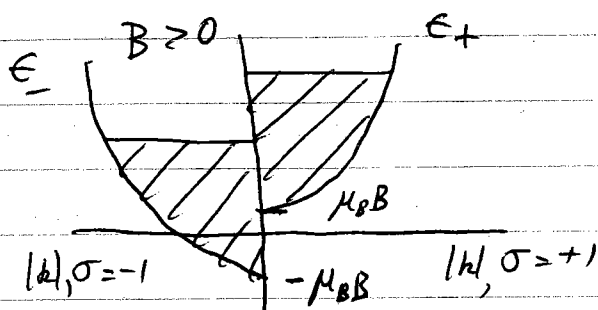
To see this, consider free electrons at  $T=0$



when  $\vec{B} = 0$ ,  $E_+(\vec{k}) = E_-(\vec{k})$

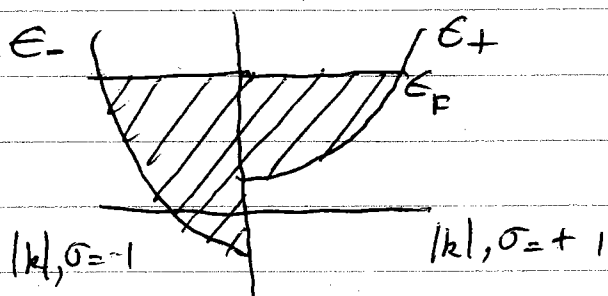
ground state occupations look as shown on the left. Equal numbers of  $\uparrow$  and  $\downarrow$  electrons  
 $m_+ = m_-$

when  $\vec{B}$  is turned on, if there were no redistribution of electron spins, the situation would look like



Clearly the system can lower its energy by transferring  $\uparrow$  electrons to  $\downarrow$  electrons.

At equilibrium the system will look like



again the two populations have the same max energy  $E_F$ .

But there are now more  $\downarrow$  electrons than  $\uparrow$  electrons

magnetization  $\frac{M}{V} = -\mu_B (m_+ - m_-) > 0$

$\frac{\vec{M}}{V}$  is parallel to  $\vec{B} \Rightarrow$  paramagnetic effect

## The calculation

Let  $g(\epsilon)$  be the density of states when  $B=0$

When  $B > 0$ , the density of states for  $\uparrow$  and  $\downarrow$  electrons are

$$\begin{aligned}g_+(\epsilon + \mu_B B) &= \frac{1}{2} g(\epsilon) &\Rightarrow g_+(\epsilon) &= \frac{1}{2} g(\epsilon - \mu_B B) \\g_-(\epsilon - \mu_B B) &= \frac{1}{2} g(\epsilon) &g_-(\epsilon) &= \frac{1}{2} g(\epsilon + \mu_B B)\end{aligned}$$

The density of  $\uparrow$  and  $\downarrow$  electrons is then

$$n_{\pm} = \int_{-\infty}^{\infty} d\epsilon g_{\pm}(\epsilon) f(\epsilon, \mu(B))$$

where  $f(\epsilon, \mu(B)) = \frac{1}{e^{(\epsilon - \mu(B))/k_B T} + 1}$

$\mu(B)$  is the chemical potential - it might depend on  $B$

- it is same for  $\uparrow$  and  $\downarrow$

We will consider only the case that

$$\mu_B B \ll \mu(B) \approx \epsilon_F$$

ie spin interaction is small compared to  $\epsilon_F$

First we will show that

$$\textcircled{1} \mu(B) \approx \mu(B=0) \left[ 1 + O\left(\frac{\mu_{BB}}{E_F}\right)^2 \right]$$

since we will work in the  $\mu_{BB} \ll E_F$  limit, we will then be able to ignore changes in  $\mu$  due to the finite  $B$ , and just use  $\mu(B=0)$ .

Proof:

Consider the total density of electrons

$$n = n_+ + n_- = \int_{-\infty}^{\infty} dE f(E, \mu(B)) [g_+(E) + g_-(E)]$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} dE f(E, \mu(B)) [g(E - \mu_{BB}) + g(E + \mu_{BB})]$$

↑ shift integration variable  $E - \mu_{BB} \rightarrow E$       ↑ shift integration variable  $E + \mu_{BB} \rightarrow E$

$$= \frac{1}{2} \int_{-\infty}^{\infty} dE g(E) [f(E + \mu_{BB}, \mu(B)) + f(E - \mu_{BB}, \mu(B))]$$

use fact that  $f(E, \mu)$  depends only on  $(E - \mu)$

$$n = \frac{1}{2} \int_{-\infty}^{\infty} dE g(E) [f(E, \mu(B) - \mu_{BB}) + f(E, \mu(B) + \mu_{BB})]$$

expand  $f$  about  $\mu(B)$  for small  $\mu_{BB}$

$$n \approx \frac{1}{2} \int d\epsilon g(\epsilon) \left[ f(\epsilon, \mu(B)) - \frac{df}{d\mu} \mu_B B + \frac{1}{2} \frac{d^2 f}{d\mu^2} (\mu_B B)^2 + \dots \right. \\ \left. + f(\epsilon, \mu(B)) + \frac{df}{d\mu} \mu_B B + \frac{1}{2} \frac{d^2 f}{d\mu^2} (\mu_B B)^2 + \dots \right]$$

where derivatives above are evaluated at  $\mu = \mu(B)$ .  
the terms linear in  $B$  cancel!

$$n \approx \int d\epsilon g(\epsilon) \left[ f(\epsilon, \mu(B)) + \frac{1}{2} \frac{d^2 f}{d\mu^2} (\mu_B B)^2 + \dots \right]$$

If we ignored the  $(\mu_B B)^2$  term the above would be

$$n = \int d\epsilon g(\epsilon) f(\epsilon, \mu(B))$$

But this is just the same ~~result~~ <sup>formula</sup> we use to compute  $n$  at  $B=0$ ! The magnetic field  $B$  appears nowhere in the above, except via  $\mu(B)$ . Since the density is physically fixed by the sample and cannot change as one varies  $B$ , we would conclude that

$$\mu(B) = \mu(0) \text{ is independent of } B!$$

conclusion

This depends on our having ignored the  $(\mu_B B)^2$  term, so we can expect

$$\mu(B) \approx \mu(0) + \frac{(\mu_B B)^2}{\epsilon_F}$$

where  $\frac{1}{\epsilon_F}$  appears on dimensional grounds.

To see this is so more explicitly, lets include the  $(\mu_B B)^2$  term and continue to calculate ...

$$n = \int d\epsilon g(\epsilon) \left[ f(\epsilon, \mu(B)) + \frac{1}{2} \frac{d^2 f}{d\mu^2} (\mu_B B)^2 \right]$$

write  $\mu(B) = \mu(B=0) + \delta\mu$  and expand in first term

$$n = \int d\epsilon g(\epsilon) \left[ f(\epsilon, \mu(B=0) + \delta\mu) + \frac{1}{2} \frac{d^2 f}{d\mu^2} (\mu_B B)^2 \right]$$

$$= \int d\epsilon g(\epsilon) f(\epsilon, \mu(B=0))$$

$$+ \int d\epsilon g(\epsilon) \left. \frac{df}{d\mu} \right|_{\mu=\mu(B=0)} \delta\mu$$

$$+ \frac{1}{2} \int d\epsilon g(\epsilon) \left. \frac{d^2 f}{d\mu^2} \right|_{\mu=\mu(B)} (\mu_B B)^2$$

The first term is just the density when  $B=0$ , i.e.  $n$ . Hence we get

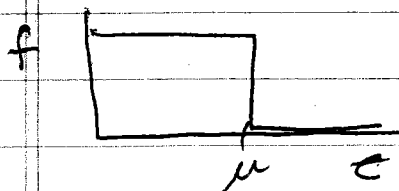
$$0 = \int d\epsilon g(\epsilon) \left. \frac{df}{d\mu} \right|_{\mu=\mu(B=0)} \delta\mu + \frac{1}{2} \int d\epsilon g(\epsilon) \left. \frac{d^2 f}{d\mu^2} \right|_{\mu=\mu(B)} (\mu_B B)^2$$

So the correction to  $\mu$  due to finite  $B$  is

$$\delta\mu = \frac{-\frac{1}{2} \int d\epsilon g(\epsilon) \left. \frac{d^2 f}{d\mu^2} \right|_{\mu=\mu(B)} (\mu_B B)^2}{\int d\epsilon g(\epsilon) \left. \frac{df}{d\mu} \right|_{\mu=\mu(B=0)}}$$



To see how big this is, consider the limit  $T \rightarrow 0$  where  $\mu(B=0) = E_F$ , and  $f$  is a step function



$$\frac{df}{d\mu} = -\frac{df}{d\epsilon} = \delta(\epsilon - \mu)$$

$$\frac{d^2f}{d\mu^2} = \frac{d^2f}{d\epsilon^2} = -\frac{d\delta(\epsilon - \mu)}{d\epsilon}$$

$$\text{so } \int d\epsilon g(\epsilon) \left. \frac{df}{d\mu} \right|_{\mu = \mu(B=0)} = g(\mu(B=0)) = g(E_F)$$

$$\int d\epsilon g(\epsilon) \left. \frac{d^2f}{d\mu^2} \right|_{\mu = \mu(B)} (\mu_B B)^2 \approx -g'(\mu(B)) (\mu_B B)^2$$

$$\delta\mu = \frac{\frac{1}{2} g'(\mu(B)) (\mu_B B)^2}{g(E_F)}$$

to lowest order, evaluate  $g'(\mu(B))$  as  $g'(E_F)$   
The difference will only give higher order corrections of  $o(\mu_B B)^4$

$$\delta\mu = \frac{g'(E_F) (\mu_B B)^2}{2g(E_F)}$$

for free electrons with  $g(\epsilon) = c\sqrt{\epsilon}$  so  
 $g'(\epsilon) = \frac{1}{2} \frac{c}{\sqrt{\epsilon}}$  we get

$$\boxed{\delta\mu = \frac{(\mu_B B)^2}{4E_F}} \quad \text{so} \quad \boxed{\mu(B) = E_F \left( 1 + \left( \frac{\mu_B B}{2E_F} \right)^2 \right)}$$

Now we compute

② Magnetization  $\frac{M}{V} = -\mu_B (m_+ - m_-) = \mu_B (m_- - m_+)$

$$\frac{M}{V} = \mu_B \int_{-\infty}^{\infty} d\epsilon f(\epsilon, \mu) [g_-(\epsilon) - g_+(\epsilon)]$$

$$= \mu_B \int d\epsilon f(\epsilon, \mu) \left[ \frac{1}{2} g(\epsilon + \mu_B B) - \frac{1}{2} g(\epsilon - \mu_B B) \right]$$

$$= \frac{1}{2} \mu_B \int d\epsilon g(\epsilon) [f(\epsilon, \mu + \mu_B B) - f(\epsilon, \mu - \mu_B B)] \text{ as before}$$

expand  $f(\epsilon, \mu \pm \mu_B B) = f(\epsilon, \mu) \pm \frac{df}{d\mu} \mu_B B$

$$\frac{M}{V} = \frac{1}{2} \mu_B \int d\epsilon g(\epsilon) \left[ 2 \frac{df}{d\mu} \mu_B B \right]$$

$$= \mu_B^2 B \int_{-\infty}^{\infty} d\epsilon g(\epsilon) \left( -\frac{\partial f}{\partial \epsilon} \right) \quad \text{since } \frac{\partial f}{\partial \mu} = -\frac{\partial f}{\partial \epsilon}$$

To lowest order in temperature  $-\frac{\partial f}{\partial \epsilon} \approx \delta(\epsilon - \mu)$  with  $\mu = \epsilon_F$

$$\boxed{\frac{M}{V} = \mu_B^2 B g(\epsilon_F)}$$

could use Sommerfeld expansion to get corrections of order  $\left(\frac{k_B T}{\epsilon_F}\right)^2$

magnetic susceptibility  $\chi = \frac{\partial(M/V)}{\partial B}$

Pauli susceptibility  $\boxed{\chi_p = \mu_B^2 g(\epsilon_F)}$   $\sim$  density of states at  $\epsilon_F$

$\epsilon_F = \frac{\hbar^2 k^2}{2m}$

For free electron gas we earlier had  $g(\epsilon_F) = \frac{3}{2} \frac{m}{\epsilon_F}$

$$\Rightarrow \boxed{\chi_p = \mu_B^2 \frac{3}{2} \frac{m}{\epsilon_F}} \quad \chi_p > 0 \Rightarrow \text{paramagnetic}$$

Compare this to classical result. Average magnetization of a single spin is

$$\langle m \rangle = \frac{\mu_B}{2} \left[ \frac{e^{-\beta \mu_B B} (+1) + e^{+\beta \mu_B B} (-1)}{e^{\beta \mu_B B} + e^{-\beta \mu_B B}} \right]$$

$$\langle m \rangle = \mu_B \tanh(\beta \mu_B B)$$

$$\frac{M}{V} = \langle m \rangle \frac{N}{V} = \mu_B n \tanh(\beta \mu_B B)$$

$$\chi = \frac{d(M/V)}{dB}$$

at low  $T \rightarrow 0$ ,  $\tanh(\beta \mu_B B) \rightarrow 1$ ,  $\frac{M}{V} \rightarrow \mu_B n$   
all spins aligned!

Compare to quantum case:

$$\frac{M}{V} = \frac{3}{2} \frac{m}{\epsilon_F} \mu_B^2 B$$

smaller than classical result by factor  $\frac{3}{2} \frac{\mu_B B}{\epsilon_F} \ll 1$

at high  $T$  ( $\beta \rightarrow 0$ )  $\tanh(\beta \mu_B B) \rightarrow \beta \mu_B B$

$$\frac{M}{V} = \frac{\mu_B^2 B m}{k_B T}$$

$$\chi = \frac{\mu_B^2 m}{k_B T} \sim \frac{1}{T}$$

Compare to quantum case - at room temp finite  $T$  corrections remain negligible and still

$$\chi_p = \mu_B^2 \frac{3}{2} \frac{m}{\epsilon_F} \quad \text{indep of } T$$

smaller than classical by factor  $\frac{3}{2} \left( \frac{k_B T}{\epsilon_F} \right) \ll 1$