Ideal Bose Gas

Bose-Einstein Condensation

Bose occupation function

\[ n(e) = \frac{1}{z^{-1} e^{\beta E} - 1} \]

We had for the density of an ideal (non-interacting) Bose gas

\[ \frac{N}{V} = \frac{1}{V} \sum_{k} \frac{1}{z^{-1} e^{\beta E(k)} - 1} = \frac{1}{(2\pi)^3} \int \frac{dk}{4\pi k^2} \frac{1}{z^{-1} e^{\beta \frac{h^2 k^2}{2m}} - 1} \]

Recall, we need \( z \leq 1 \) for the occupation number at \( E(k=0) = 0 \) to remain positive \( M(0) \geq 0 \)

\[ M(0) = \frac{1}{2 - z} = \frac{z}{1 - z} \Rightarrow z \leq 1 \], \( z = e^{\beta \mu} \Rightarrow M \leq 0 \)

Substitute variables \( y = \frac{\beta \frac{h^2 k^2}{2m}}{} \Rightarrow k = \sqrt{\frac{2my}{\beta h^2}} \)

\[ dk = \sqrt{\frac{2my}{\beta h^2}} \frac{dy}{y} \]

\[ \Rightarrow \frac{N}{V} = \left( \frac{2m}{\beta h^2} \right)^{3/2} \frac{1}{4\pi} \int \frac{dy}{y} \frac{y^{1/2}}{z^{-1} e^y - 1} \]

\[ \frac{N}{V} = \frac{1}{2^3} g_{3/2}(z) \text{ where } A = \left( \frac{\hbar^2}{2\pi mk_BT} \right)^{1/2} \text{ thermal wavelength} \]

\[ g_{3/2}(z) = \frac{2}{\sqrt{\pi}} \int \frac{dy}{y} \frac{y^{1/2}}{z^{-1} e^y - 1} \]
Consider the function

\[ g_{3/2}(z) = \frac{2}{\sqrt{\pi}} \int_0^\infty dy \frac{y^{1/2}}{z^{-1} e^y - 1} = z + \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} + \ldots \]

\( g_{3/2}(z) \) is a monotonic increasing function of \( z \) for \( z \leq 1 \)

As \( z \to 1 \), \( g_{3/2}(z) \) approaches a finite constant

\[ g_{3/2}(1) = 1 + \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} + \ldots \approx \zeta(3/2) \approx 2.612 \]

(\( \zeta \) is the Riemann zeta function)

We can see that \( g_{3/2}(1) \) is finite as follows:

\[ g_{3/2}(1) = \frac{2}{\sqrt{\pi}} \int_0^\infty dy \frac{y^{1/2}}{e^y - 1} \]

as \( y \to \infty \) the integral converges. Integral is largest at smallest \( y \)

(recall smallest \( y \) corresponds to low energy where \( m(m) \) is largest)

For small \( y \), we can approximate \( \frac{1}{e^y - 1} \approx \frac{1}{y} \)

\[ \int_0^{y^*} dy \frac{y^{1/2}}{e^y - 1} \approx \int_0^{y^*} dy \frac{1}{y^{1/2}} = \left. 2 \sqrt{y} \right|_0^{y^*} \]

So, we see the integral also converges at its lower limit \( y \to 0 \).

\[ \zeta(3/2) \]

\[ g_{3/2}(z) \]

\[ 0 \quad 1 \quad z \]
So we conclude

\[ n = \frac{N}{V} = \frac{9.3/2(7)}{2^8} \leq \frac{9.3/2(1)}{2^3} = \frac{2.612}{2^3} = 2.612 \left( \frac{2\pi mk_B T}{\hbar^2} \right)^{3/2} \]

But we now have a contradiction!

For a system with fixed density of bosons \( n \), as \( T \) decreases we will eventually get to a temperature below which the above inequality is violated.

The temperature is

\[ T_0 = \left( \frac{m}{2\pi k} \right)^{1/3} \frac{\hbar^2}{2\pi mk_B} \]

Solution to the paradox:

when we made the approx \( \frac{1}{V} \sum k \rightarrow \frac{1}{(2\pi)^3} \int dk \frac{4\pi k^2}{(2\pi)^3} \)

we gave a weight \( \frac{4\pi k^2}{(2\pi)^3} \) to states with wavevector \( |k| \).

This gives zero weight to the state \( k = 0 \), i.e. to the ground state. But as \( T \) decreases, more and more bosons will occupy the ground state, as it has the lowest energy. This when we approx the sum by an integral, we should treat the ground state separately

\[ \frac{1}{V} \sum \frac{n(E(k))}{k} \approx \frac{n(0)}{V} + \frac{1}{(2\pi)^3} \int dk \frac{4\pi k^2}{(2\pi)^3} n(E(k)) \]

\[ \left| \frac{n(0)}{V} \right| = \frac{1}{V} \sum \frac{n(E(k))}{k} \left| \frac{\partial n(E(k))}{\partial E(k)} \right| \left( k^2 \right) \]

This term is important when \( n(0)/V \) stays finite as \( V \rightarrow \infty \), i.e. a macroscopic fraction of bosons occupy the ground state.
\[ M = \frac{N}{V} = \frac{n(0)}{V} + \frac{g_{3/2}(z)}{\lambda^3} \]

\[ M = m_0 + \frac{g_{3/2}(z)}{\lambda^3} \quad \text{where } m_0 = \frac{n(0)}{V} \text{ density of bosons in ground state} \]

For a system with fixed \( m \), at higher \( T \) one can always choose \( z \) so that \( m = \frac{g_{3/2}(z)}{\lambda^3} \) and \( m_0 = 0 \).

But when \( T < T_c \) it is necessary to have \( m_0 > 0 \).

Using \( n(0) = \frac{z}{1-z} \)

we can write above as

\[ M = \frac{z}{1-z} \frac{1}{V} + \frac{g_{3/2}(z)}{\lambda^3} \]

For \( T > T_c \), we will have a solution to the above for some fixed \( z < 1 \). In thermodynamic limit \( V \to \infty \), the first term will then vanish, i.e., the density of bosons in the ground state vanishes.

As \( T \to T_c \), \( z \to 1 \) and the first term \( \left( \frac{z}{1-z} \right) \left( \frac{1}{V} \right) \) stays finite to give the additional needed density at \( T < T_c \):

\[ \frac{z}{1-z} \frac{1}{V} = m_0 = m - \frac{g_{3/2}(1)}{\lambda^3} \]
To define the Bose-Einstein transition temperature below which the system develops a finite density of particles in the ground state $N_0$. $N_0$ is also called the condensate density. The particles in the ground state are called the condensate.

$$Z(T) \rightarrow 1 \text{ as } T \rightarrow T_c, \quad Z(T) = 1 \text{ for } T \leq T_c, \quad \mu(T) = 0$$

For $T < T_c$:

$$N_0(T) = m - g_{3/2}(1) = m - 2.612 \left( \frac{2\pi m k_B T}{\hbar^2} \right)^{3/2}$$

$$N_0(T) = m \left( 1 - \left( \frac{T}{T_c} \right)^{3/2} \right)$$

The condensate density vanishes continuously as $T \rightarrow T_c$ from below.

At $T = 0$, all bosons are in condensate.
At $T > T_c$, all bosons are in the "normal state".
At $0 < T < T_c$, a macroscopic fraction of bosons are in the condensate, while the remaining fraction are in the normal state, call it the "mixed state".