Having outlined what we might expect from the Ising model, we would now like to compute properties and see what happens!

However, an exact solution is not in general possible. Exact solutions to Ising model exist in:
- $d=1$ dimension - we will do this later
- $d=2$ dimension - famous solution by Onsager
- $d=3$ dimensions, the best one has very accurate numerical simulations - no exact solution.

$\Rightarrow$ Approximate Solution

- Mean Field or Curie-Weiss Molecular Field Approximation

$$H = -J \sum_{\langle i,j \rangle} S_i S_j - h \sum_i S_i$$

Consider spin $S_i$. Approximate the interaction of $S_i$ with its neighbor $S_j$ as an interaction with the thermal average value of $\langle S_j \rangle \equiv m$.

Instead of $S_i$ seeing the specific $S_j$ that vary from config to config, we say $S_i$ sees only the effective average value of $S_j$ - which is the same value $m$ for all sites $j$. $\langle S_j \rangle = m = \frac{1}{N} \sum_i S_i$

$$H_{\text{MF}} = -J \sum_{\langle i,j \rangle} S_i m - h \sum_i S_i$$

(sum over bonds)
\[ H_{MF} = -\frac{z J}{2} \sum_i S_i \cdot m - \hbar \sum_i S_i \]

\[ = -\left( \frac{z J}{2} m + \hbar \right) \sum_i S_i \]

where \( z \) is the "coordination number" — the number of nearest neighbors of site \( i \). For a simple (3d) cubic lattice, \( z = 6 \). For a (2d) square lattice, \( z = 4 \).

In this approach, the interaction of \( S_i \) with its neighbors is just like the interaction of \( S_i \) with an additional average magnetic field \( \frac{z J}{2} m \) — hence the origin of the name "mean field" approach.

To complete the approach, we need to self-consistently compute \( m \) using \( H_{MF} \) and self-consistently solve for \( m \) from the resulting equation.

\[ H_{MF} = \sum_i H_{MF}^{(i)} \quad \text{where} \quad H_{MF}^{(i)} = -\left( \frac{z J}{2} m + \hbar \right) S_i \]

we have non-interacting spins via MF approach.

\[ \Rightarrow \langle S \rangle = m = \frac{\sum S e^{-\beta H_{MF}^{(i)}}}{\sum S e^{-\beta H_{MF}^{(i)}}} \]

\[ = \frac{e^{\beta \left( \frac{z J}{2} m + \hbar \right)} e^{-\beta \left( \frac{z J}{2} m + \hbar \right)}}{e^{\beta \left( \frac{z J}{2} m + \hbar \right)} + e^{-\beta \left( \frac{z J}{2} m + \hbar \right)}} \]

\[ m = \tanh \left[ \beta \left( \frac{z J}{2} m + \hbar \right) \right] \]

Solve to get \( m(T, \hbar) \)

Note \( m(T, \hbar) = -m(T, -\hbar) \) as expected.
\[ m = \tanh \left[ \frac{\beta z J m + \beta k}{2} \right] \]

**Consider first \( k = 0 \)**

\[ m = \tanh \left[ \frac{\beta z J m}{2} \right] \text{ could be solved graphically} \]

Since \( \tanh x = x - \frac{1}{3} x^3 + O(x^5) \) we see that

for \( \frac{\beta z J}{2} < 1 \), the only solution will be \( m = 0 \).

However, for \( \frac{\beta z J}{2} > 1 \) there are two additional solutions \( m = \pm m_0 \)

\[ \tanh \left( \frac{\beta z J m}{2} \right) \]

\[ \beta z J > 1 \]

\[ \beta z J < 1 \]

\[ \Rightarrow \text{critical temperature} \quad k_B T_c = \frac{z J}{2} \]

\[ T > T_c \Rightarrow m = 0 \]

\[ T < T_c \Rightarrow m = \pm m_0 \quad m = 0 \text{ is unstable solution} \]
For $T < T_c$, $m=0$ is unstable $m = \pm m_0$ are the equilibrium solutions. To see this:

$$m = \tanh \left( \frac{\beta \varepsilon J m + \beta \chi}{2} \right)$$

$$\chi = \frac{1}{\beta} \tanh^{-1} m - \frac{Z J m}{2}$$

$$\int_{m}^{0} \frac{df}{dm} \ dm = \int_{0}^{m} h \ dm' \Rightarrow f(m, T) = f(0, T) + \int_{0}^{m} h(m') \ dm'$$

Therefore we can plot the free energy $f(m, T) - f(0, T)$.

$\int h(m') \ dm'$ is the negative of the shaded area shown to the left. We see this area increases in magnitude until $m = m_0$, and then decreases in magnitude as $m$ exceeds $m_0$ (since the curves cross at $m_0$).

So $f(m_0, T) < f(0, T)$, $m_0$ gives the min. of the free energy and so is the equilibrium solution.
For \( T > T_c \), the situation looks like

\[
\begin{align*}
\text{now } \int h(m)dm & \text{ is the positive of the area shown} \\
& \text{to the left - it increases monotonically as } m \text{ increases}
\end{align*}
\]

so the free energy looks like

\[
\begin{align*}
g(m,T) &= f(0,T) \\
\Rightarrow m = 0 & \text{ is minin of } g(m,T)
\end{align*}
\]

\[
\begin{align*}
g(h=0,T) &= \min_m g(m,T) \\
\Rightarrow m = 0 & \text{ is liquid state}
\end{align*}
\]

Recall - the plots of \( f(m,T) \) shown above for \( T > T_c \)
and \( T < T_c \) are exactly the same as we saw in
discussing the van der Waals theory of the liquid-gas
phase transition!
We can examine these points analytically. If we consider behavior near \( T_c \) where \( m \) is small, this analysis will introduce the critical exponents \( \delta, \beta, \gamma \) that characterize the critical point at \( (T_c, h = 0) \):

\[
m = \tanh \left( \frac{\beta}{2} \frac{k_B T}{m + \beta} \right)
\]

Use \( \frac{\beta}{2} = k_B T_c \), \( \tanh x \approx x - \frac{1}{3} x^3 \) for small \( x \), for small \( h \), near \( T_c \) where \( m \) is small, expand the \( \tanh \):

\[
m = \left( \frac{T_c}{T} m + \frac{h}{k_B T} \right) \left( 1 - \frac{1}{3} \left( \frac{T_c}{T} \right)^3 m^3 \right)
\]

for small \( \frac{h}{k_B T} \):

\[
m = \left( \frac{T_c}{T} m + \frac{h}{k_B T} \right) \left( 1 - \frac{1}{3} \left( \frac{T_c}{T} \right)^3 m^3 \right) = \frac{h}{k_B T} \left( 1 - \left( \frac{T_c}{T} \right)^2 m^2 \right)
\]

\[
h = k_B T \sqrt{\frac{m (1 - \frac{T_c}{T}) + \frac{1}{3} \left( \frac{T_c}{T} \right)^3 m^3}{1 - \left( \frac{T_c}{T} \right)^2 m^2}}
\]

At \( T = T_c \) critical isotherm:

\[
h = k_B T \left( \frac{1}{3} m^3 \propto m^\delta \quad \delta = 3 \right)
\]

or \( m \propto h^{1/3} \):

\[
h \quad \propto \quad \frac{1}{3} m^3
\]

\( T \quad \propto \quad \frac{1}{3} m^3 \)

\( T_c \quad \propto \quad m \propto h^{1/3} \)
2. At $h=0$ on coexistence line

\[
\left(1 - \frac{T_c}{T}\right) m + \left[ \frac{1}{3} \left( \frac{T_c}{T} \right)^3 + \left(1 - \frac{T_c}{T}\right) \left( \frac{T_c}{T} \right)^2 \right] m^3 = 0
\]

as $T \to T_c^-$, \( \left(1 - \frac{T_c}{T}\right) + \frac{1}{3} m^2 = 0 \)

\[
m = \pm \sqrt[3]{\frac{3(T_c - T)}{T}}
\]

Define $t = \frac{T_c - T}{T_c}$

\[m \propto \pm \sqrt[3]{3t} \propto t^{\beta} \quad \beta = \frac{1}{2}
\]

3. At $h=0$ on coexistence line as $T \to T_c$

\[
\frac{\partial h}{\partial m} = k_B T \left\{ \left(1 - \frac{T_c}{T}\right) + 3 \left(1 - \frac{T_c}{T}\right) \left( \frac{T_c}{T} \right)^2 \right\} m^2 \frac{\partial m}{\partial \chi} \nabla \frac{\partial m}{\partial \chi} = k_B T \left\{ \left(1 - \frac{T_c}{T}\right) + m^2 \right\}
\]

As $T \to T_c^+$ from above, $m = 0$

\[
\frac{\partial h}{\partial m} = k_B T \left(1 - \frac{T_c}{T}\right) = k_B (T - T_c)
\]

magnetic susceptibility
\[
\frac{\partial m}{\partial \chi} = \chi^+ = \frac{1}{k_B (T - T_c)} \propto \frac{1}{T - T_c}, \quad \gamma = 1
\]

Note: at high temp $T \gg T_c$, $\chi \sim \frac{1}{T}$ just like in Curie paramagnetism. Hence we say the $T > T_c$ phase is paramagnetic.
As $T \to T_c^-$ from below, $m^2 = 3 \frac{(T_c - T)}{T}$

$\Rightarrow \frac{2k}{\partial m} = k_B T \left( 1 - \frac{T_c}{T} \right) + 3 \frac{(T_c - T)}{T}$

$= 2k_B (T_c - T)$

$\frac{\partial m}{\partial \lambda} \frac{\lambda}{m} = \frac{l}{2k_B (T_c - T)} \propto \frac{1}{1+\gamma} \text{ for } \gamma = 1$

Also $\lim_{T \to T_c} \left( \frac{X^+}{X^-} \right) = \frac{2k_B (T_c - T)}{k_B (T_T - T)} = 2 \text{ amplitude ratio}$

Our results here are identical to what we found for the ideal model of liquid-gas transition, if we make the identifications:

$L \leftrightarrow \delta p - \delta p^*(T) \text{ distance from coexistence curve}$

$m \leftrightarrow \delta v - \delta v_0$

Free energy

$g(m, T) = \int f(m') dm'$

$\Rightarrow g(m, T) - g(0, T) = k_B T \left\{ \frac{1}{2} \left( 1 - \frac{T_c}{T} \right) m^2 + \frac{1}{4} m^4 \right\}$

Coefficient of $m^2$ term vanishes at $T_c$, goes negative below $T_c \to$ minimum of $g(m, T)$ changes from $m = 0 \to m = \pm m_0(T)$

$g(\delta h = 0, T) = \min_m f(m, T) \Rightarrow \min_m g$ gives equilibrium state
\[ h = k_b T \left\{ (1 - \frac{T_c}{T}) m + \frac{1}{3} m^3 \right\} \]

For \( T < T_c \) we should "fix" the \( h(m) \) curve with a Maxwell construction at \( h = 0 \) from \( m = -m_0(T) \) to \( m = +m_0(T) \).

Similarly, \( f(m, T) \) with Maxwell construction looks like

\[ f \]

\[ T < T_c \]

Gibbs

\[ g(h) \] has discontinuous derivative at \( h = 0 \) for \( T < T_c \).