Note: The mean field approx is *exact* in the limit that every spin interacts with every other spin (not just nearest neighbors). Then

$$\mathcal{H} = -\tilde{J} \sum_{i,j} s_i s_j - h \sum_i s_i$$

$$= -\tilde{J} \sum_i s_i \left(\sum_j s_j \right) - h \sum_i s_i$$

$$= -\tilde{J} \sum_i s_i \, N m - h \sum_i s_i$$

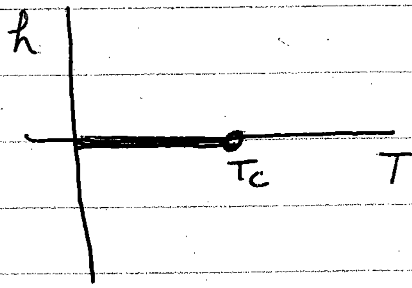
$$\mathcal{H} = -\left(\frac{z J}{2} m + h \right) \sum_i s_i$$

where we took $J \equiv \frac{2}{z} \frac{\tilde{J}}{N}$. For infinite range coupling model, need to take coupling $J \propto \frac{1}{N}$ so that total energy scales with $E \propto N$ as desired.

In the above, $m[s_i] = \frac{1}{N} \sum_j s_j$ *depends* on the config $\{s_i\}$, however it is the same for every spin s_i

Landau's Theory of phase transitions

Ising model



order parameter m – $\begin{cases} \text{zero} & T > T_c \\ \text{non-zero} & T < T_c \end{cases}$

ordering field h

applying $h \neq 0$ reduces symmetry of Hamiltonian, induces $m \neq 0$.

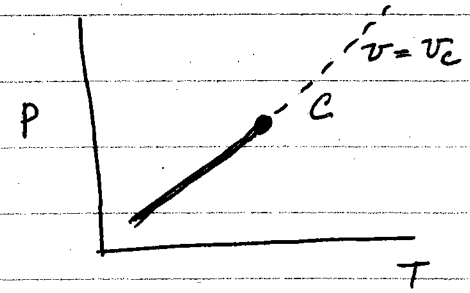
For ordering field $h = 0$, Hamiltonian has higher symmetry. A finite order parameter $m \neq 0$ breaks this symmetry!

For $h = 0$, 2^{nd} order phase transition at T_c such that $m = 0$ for $T > T_c$ $\Rightarrow$ thermodynamic state has full symmetry of $\mathcal{H}$. When $T < T_c$ the order parameter becomes finite $m \neq 0$ $\rightarrow$ thermodynamic state breaks symmetry of $\mathcal{H}$. Spontaneous symmetry breaking

For $T < T_c$ varying the ordering field h through zero results in a discontinuous jump in the order parameter – 1^{st} order transition line

~~At~~ the 2^{nd} order transition at T_c, that ends the 1^{st} order transition line, m goes to zero continuously as $T \to T_c^-$.

For liquid-gas transition



1st order line does not have any particular symmetry with respect to the natural thermodynamic variables

dashed line is critical isochore – line of constant $v = v_c$ smoothly extends from 1st order transition line.

Define ordering field δp as distance from critical isochore. Define order parameter $\delta v = v - v_c$ as difference in specific volume (or density) from critical value.

Landau methodolgy

① Given a physical system with a phase transition, first identify the order parameter m – a quantity that vanishes in the high T disordered phase, and is non zero in the low T ordered phase. Often this can be the hard part! cf. spin glass problem

② Near the 2nd order critical pt, the order parameter is small. ⇒ expand the Helmholtz free energy $f(m,T)$ in a Taylor series in m, keeping all terms which have the appropriate symmetry of the problem.

For Ising model

$$f(m,T) \simeq f_0 + am^2 + bm^4 + \cdots$$

only even powers of m appear since $\mathcal{H} = -J\sum_{\langle ij \rangle} s_i s_j$ is symmetric under $\{s_i\} \to \{-s_i\}$.

For liquid - gas transition

$$f(\delta v,T) \simeq f_0 + a\delta v^2 + b\delta v^3 + c\delta v^4 + \cdots$$

no symmetry of $\mathcal{H}$ to rule out odd powers of order parameter δv. - but still no linear δv term since $f(\delta v=0,T)$ must be a minimum when $T>T_c$. - or equivalently, if there was a δv linear term in $f(\delta v,T)$, it would mean we were not properly expanding about critical isochore.

But we saw that by the trick of transformation of variables, $\delta v = \delta v_0 + u$, we can effectively eliminate the δv^3 term and make the problem look just like the Ising model.

$$f(m,T) = f_0(T) + a(T)m^2 + b(T)m^4 + \cdots$$

↑ ignore higher order terms

Stability $\Rightarrow b(T) > 0$, $f(m,T)$ must have global minimum

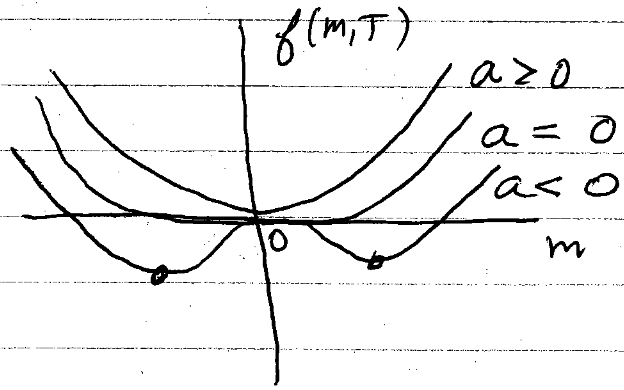
State of system is obtained by minimizing $f(m,T)$ with respect to m. Or equivalently, Gibbs free energy is

$$g(h,T) = \min_m \left[f(m,T) - hm \right]$$

on the 1st order line the ordering field $h = 0$

$$\Rightarrow g(0,T) = \min_m \left[f(m,T) \right]$$

$\Rightarrow$ 2nd order critical point occurs when $a(T) = 0$



When $a \geq 0$ then $m=0$ minimizes $f(m,T)$
$\Rightarrow$ thermodynamic state has symmetry of H

When $a<0$, then $m = \pm m_0$ minimizes $f(m,T)$
$\Rightarrow$ thermodynamic state breaks symmetry

minimum of $f(m,T)$ increases <u>continuously</u> from $m=0$ as a decreases below zero.