Note: The mean field approx is exact in the limit that every spin interacts with every other spin (not just nearest neighbors). Then

\[ H = -J \sum_{i,j} \sigma_i \sigma_j - h \sum_i \sigma_i \]

\[ = -J \sum_i \sigma_i (\sum_j \sigma_j) - h \sum_i \sigma_i \]

\[ = -J \sum_i \sigma_i Nm - h \sum_i \sigma_i \]

\[ H = -(\frac{2J}{N}m + h) \sum_i \sigma_i \]

where we took \( J = \frac{\tilde{J}}{N} \). In infinite range coupling model, need to take coupling \( J < \frac{1}{N} \) so that total energy scales with \( E \propto N \) as desired.

In the above, \( m[\sigma_i] = \frac{1}{N} \sum \sigma_i \) depends on the config \( \{\sigma_i\} \), however it is the same for every spin \( \sigma_i \).
Landau's Theory of phase transitions

Ising model

\[ h \]

- order parameter \( m \) - \( \begin{cases} \text{zero} & T > T_c \\ \text{non-zero} & T < T_c \end{cases} \)
- ordering field \( h \)

Applying \( h \neq 0 \) reduces symmetry of Hamiltonian, induces \( m \neq 0 \).

For ordering field \( h = 0 \), Hamiltonian has higher symmetry. A finite order parameter \( m \neq 0 \) breaks this symmetry.

For \( h = 0 \), 2nd order phase transition at \( T_c \) such that \( m = 0 \) for \( T > T_c \) \( \Rightarrow \) thermodynamic state has full symmetry of \( H \). When \( T < T_c \), the order parameter becomes finite \( m \neq 0 \) \( \Rightarrow \) thermodynamic state breaks symmetry of \( H \), spontaneous symmetry breaking.

For \( T < T_c \), varying the ordering field \( h \) through zero results in a discontinuous jump in the order parameter - 1st order transition line.

Note the 2nd order transition at \( T_c \), that ends the 1st order transition line, \( m \) goes to zero continuously as \( T \to T^- \).
For liquid-gas transition

![Diagram of P vs T with critical isochore line](image)

1st order line does not have any particular symmetry with respect to the natural thermodynamic variables.

dashed line is critical isochore - line of constant \( V = V_c \)

smoothly extends from 1st order transition line.

Define ordering field \( S \) as distance from critical isochore. Define order parameter \( \delta V = V - V_c \) as difference in specific volume (or density) from critical value.

Landau methodology

1. Given a physical system with a phase transition, first identify the order parameter \( m \) - a quantity that vanishes in the high T disordered phase, and is non-zero in the low T ordered phase. Often this can be the hard part! cf. spin glass problem.

2. Near the 2nd order critical pt, the order parameter is small. \( \Rightarrow \) expand the Helmholtz free energy \( f( M, T ) \) in a Taylor series in \( m \), keeping all terms which have the appropriate symmetry of the problem.
For Ising model

\[ f(m,T) = f_0 + a m^2 + b m^4 + \ldots \]

only even powers of \( m \) appear since \( H = -J \sum_i s_i s_j \)

is symmetric under \( s_i \rightarrow -s_i \).

For liquid - gas transition

\[ f(\delta v, T) = f_0 + a \delta v^2 + b \delta v^3 + c \delta v^4 + \ldots \]

no symmetry of \( H \) to rule out odd powers of order parameter \( \delta v \) - but still no linear \( \delta v \) term

since \( f(\delta v=0, T) \) must be a minimum when \( T \geq T_c \)

- or equivalently, if there was a \( \delta v \) linear term

\( f(\delta v, T) \), it would mean we were not properly expanding about critical isochore.

But we saw that by the trick of transformation of variables, \( \delta v = \delta v_0 + u \), we can effectively eliminate the \( \delta v^3 \) term and make the problem look first like the Ising model.
\[ f(m, T) = f_0(T) + a(T) m^2 + b(T) m^4 + \ldots \]

Ignore higher order terms.

Stability \( \Rightarrow b(T) > 0 \) \( \Rightarrow f(m, T) \) must have global minima.

State of system is obtained by minimizing \( f(m, T) \) with respect to \( m \), or equivalently, Gibbs free energy is

\[ g(-h_{-T}) = \min_{m} \left[ f(m, T) - h m \right] \]

On the 1st order line the ordinary field \( h = 0 \)

\[ g(0, T) = \min_{m} \left[ f(m, T) \right] \]

\( \Rightarrow \) 2nd order critical point occurs when \( a(T) = 0 \)

\begin{align*}
  &\text{When } a > 0 \text{ then } \\
  &m = 0 \text{ vanishes } f(m, T) \\
  &\Rightarrow \text{ thermodynamic state has symmetry of } H
\end{align*}

\begin{align*}
  &\text{When } a < 0 \text{ then } m = \pm m_0 \\
  &m = 0 \text{ vanishes } f(m, T) \\
  &\Rightarrow \text{ thermodynamic state breaks symmetry}
\end{align*}

\[ \text{min} \text{ of } f(m, T) \text{ increases continuously from } m = 0 \as a decreases below zero} \]