Expanding near T_c to lowest orders,

$$b(T) \approx b(T_c) \equiv b_0 \quad \text{a constant}$$

$$a(T) \simeq a_0[T - T_c] \quad a_0 \text{ a constant}$$

① <u>Behavior of order parameter near T_c</u>

$T < T_c$ minimize $f(m,T)$

$$\Rightarrow \quad 2a\, m + 4b\, m^3 = 0$$

$$2a + 4bm^2 = 0 \quad \text{for } m \neq 0$$

$$m^2 = -\frac{a}{2b}$$

$$\boxed{m_0 = \pm\sqrt{\frac{a_0(T_c - T)}{2b_0}}} \propto |t|^\beta \quad \boxed{\beta = 1/2}$$

$$t = \left(\frac{T_c - T}{T_c}\right)$$

same β as found earlier

② <u>$h(m)$ curve at critical isotherm $T = T_c$</u>

$$g(h, T_c) = \min_m \left[f(m,T) - hm \right]$$

$$= \min_m \left[f_0 + b_0 m^4 - hm \right] \quad a = 0 \text{ at } T_c$$

$$\Rightarrow \quad 4b_0 m^3 - h = 0 \quad \Rightarrow \quad \boxed{h = 4b_0 m^3}$$

$$h \propto m^\delta \quad \boxed{\delta = 3} \quad \text{same as before}$$

③ susceptibility $\chi = \frac{\partial m}{\partial h}$ at $h = 0$

$$g(h,T) = \min_m \left[f(m,T) - h m \right]$$

$$\Rightarrow \quad 2am + 4bm^3 = h \qquad \text{"equation of state"}$$

$$\chi^{-1} = \frac{\partial h}{\partial m} = 2a + 12bm^2$$

$$\chi = \frac{1}{2a + 12bm^2}$$

For $T > T_c$, $h = 0 \Rightarrow m^2 = 0$

$$\boxed{\chi^+ = \frac{1}{2a}} = \frac{1}{2a_0(T - T_c)} \propto \frac{1}{|t|^{\gamma'}}, \quad \boxed{\gamma' = 1}$$

For $T < T_c$, $h = 0 \Rightarrow m^2 = m_0^2 = \frac{-a}{2b} = \frac{a_0(T_c - T)}{2b_0}$

$$\chi^- = \frac{1}{2a_0(T - T_c) + \frac{12 b_0 a_0}{2b_0}(T_c - T)}$$

$$\boxed{\chi^- = \frac{1}{4a_0(T_c - T)}} \propto \frac{1}{|t|^{\gamma}} \qquad \boxed{\gamma = 1}$$

$$\boxed{\lim_{T \to T_c} \frac{\chi^+}{\chi^-} = 2}$$ amplitude ratio

all same as before

④ <u>specific heat at $h=0$</u> along 1st order transition line

from ① we have $m_0^2 = \frac{-a}{2b}$ $T<T_c$, $m_0^2 = 0$ $T>T_c$

$$\Rightarrow \quad g(h=0,T) = f(m_0,T) = f_0(T), \quad T>T_c$$

$$= f_0(T) + a\left(\frac{-a}{2b}\right) + b\left(\frac{-a}{2b}\right)^2, \quad T<T_c$$

$T<T_c$:
$$f(m_0,T) = f_0(T) - \frac{a^2}{2b} + \frac{a^2}{4b} = f_0(T) - \frac{a^2}{4b}$$

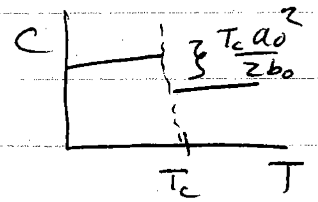
$$= f_0(T) - \frac{a_0^2}{4b_0}(T-T_c)^2$$

<u>specific heat</u>

$$s = -\frac{\partial g}{\partial T} \quad \Rightarrow \quad C = T\left(\frac{\partial s}{\partial T}\right)_{h=0} = -T\frac{\partial^2 g}{\partial T^2}$$

$$C = -T\frac{d^2}{dT^2} f(m_0(T),T)$$

$$= \begin{cases} -T\dfrac{d^2 f_0}{dT^2} & T>T_c \\[2ex] -T\dfrac{d^2 f_0}{dT^2} + \dfrac{T a_0^2}{2b_0} & T<T_c \end{cases}$$



$$\Rightarrow \quad C(T\to T_c^-) - C(T\to T_c^+) = \frac{T_c a_0^2}{2b_0}$$

jump in specific heat at T_c

The piece $\frac{\partial^2 f_0}{\partial T^2}$ is the non singular piece of the specific heat. f_0 is the same as the "reference" free energy we used earlier when integrating the ~~equation~~ of state in the mean field or the van der Waals approx.

We can define a critical exponent α for the specific heat by $C \propto |t|^\alpha$, or

$$\alpha = \lim_{t \to 0} \left[\frac{\ln C}{\ln |t|} \right]$$

For Landau theory this gives $\boxed{\alpha = 0}$

Summary: Landau theory = mean field theory

$$\left.\begin{array}{lll}
h = 0, & m_0(T) \sim |t|^\beta & \underline{\beta = 1/2} \\
T = T_c, & h(m) \propto m^\delta & \underline{\delta = 3} \\
h = 0, & \chi(T) \propto \frac{1}{|t|^\gamma} & \underline{\gamma = 1} \\
 & \lim_{t \to 0} \frac{\chi^+}{\chi^-} = 2 &
\end{array}\right\} \text{mean field critical exponents}$$

$$h = 0, \quad C(T) \propto |t|^\alpha \quad \underline{\alpha = 0}$$

exponent values in mean field approx are indep of dimension d.

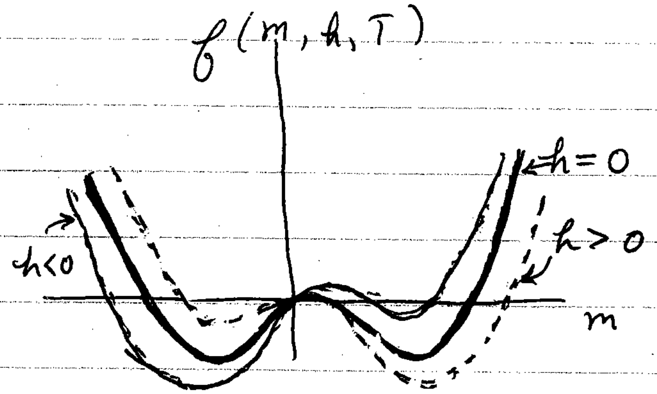
From exact solution of 2D Ising model

$\delta = 15$ $\quad \beta = 1/8$, $\gamma = 7/4$, $\underset{\text{log divergence}}{\alpha = 0}$, $\nu = 1$ $\qquad C \propto \ln|t|$

Landau theory of 1st order transition

For $T < T_c$, $h \neq 0$

$$g(h,T) = \min_m \left[f(m,T) - hm \right] \equiv \min_m \left[f(m,h,T) \right]$$



as h goes smoothly through zero, the value of m that minimizes $f(m,h,T)$ jumps discontinuously from $+m_0$ to $-m_0$.

2nd order transition – order parameter goes continuously to zero
1st order transition – order parameter jumps discontinuously

Note: Landau theory ≡ mean field theory
gives the same values of the critical exponents independent of dimension d, and number of components of spin n.

For n-component spins with $\vec{m} = \frac{1}{N} \sum_i \vec{s}_i$

$$f(\vec{m}, T) = f_0 + a|\vec{m}|^2 + b|\vec{m}|^4 + \cdots$$

everything comes out the same!

But can get some interesting new behaviors by doing other thing

① $f(m,T) = f_0 + a\, m^2 - b\, m^4 + c\, m^6$

$b > 0 \Rightarrow$ quartic term is negative
need m^6 term to give stability

This describes a tricritical point where a line of 1st order transitions becomes a line of 2nd order transitions

② put in spatially vary terms: ex: a superconductor in an applied magnetic field. Order parameter is condensate wavefunction $\psi(\vec{r})$.

magnetic vector potential

$$f(\psi, T) = f_0 + a|\psi|^2 + b|\psi|^4 + c\left|(\vec{\nabla} + i\vec{A})\psi\right|^2$$

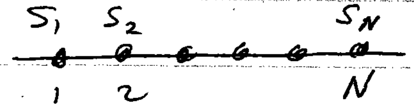
minimize wrt ψ to get Abrikosov vortex lattice.

$\vec{\nabla} \times \vec{A} = \vec{B}$ magnetic field

kinetic energy of supercurrents

Ising model in 1-dimension

$h=0$ for simplicity



$$\mathcal{H} = -J\sum_{i=1}^{N} s_i s_{i+1}$$

Define $\sigma_i = s_i s_{i+1}$, $i=1,\cdots,N-1$

$$\sigma_i = \pm 1$$

$$\mathcal{H} = -J\sum_{i=1}^{N-1}\sigma_i \qquad s_1 s_j = \prod_{i=1}^{j-1}\sigma_i = (s_1 s_2)(s_2 s_3)\cdots(s_{j-1}s_j)$$
$$= s_1 s_2^2 s_3^2 \cdots s_{j-1}^2 s_j$$
$$= s_1 s_j$$

For every set of $\{\sigma_i\}_{i=1}^{N-1}$ there are 2 possible spin configurations depending on whether $s_1 = +1$ or -1

For a given value of s_1, then

$$s_j = \frac{1}{s_1}\prod_{i=1}^{j-1}\sigma_i$$

So

$$Z = \sum_{\{s_i\}} e^{\beta J\sum_{i=1}^{N} s_i s_{i+1}} = \underbrace{2}_{\text{two values for } s_1}\sum_{\{\sigma_i\}} e^{\beta J\sum_{j=1}^{N-1}\sigma_j} = 2\prod_{i=1}^{N-1}\sum_{\sigma_i=\pm1} e^{\beta J\sigma_i}$$

$$Z = 2\left[\sum_{\sigma=\pm1} e^{\beta J\sigma}\right]^{N-1} = 2\left[2\cosh\beta J\right]^{N-1}$$

Gibbs free energy

$$G(h=0,T) = -k_B T \ln Z = -k_B T \ln 2 - k_B T(N-1) \ln(2\cosh\beta J)$$

$$g = \lim_{N\to\infty} \frac{G}{N} = -k_B T \ln(2\cosh\beta J)$$

entropy $s = -\left(\frac{\partial g}{\partial T}\right)_{h=0}$ specific heat at const $h=0$ $C = T\left(\frac{\partial s}{\partial T}\right)_{h=0}$

$$= -T\left(\frac{\partial^2 g}{\partial T^2}\right)$$

$$s = k_B \ln(2\cosh\beta J) + \frac{k_B T}{2\cosh(\beta J)} 2 \frac{\partial}{\partial T}\left[\cosh(\beta J)\right]$$

$$= k_B \ln(2\cosh\beta J) + \frac{k_B T}{\cosh(\beta J)} \sinh(\beta J) J \frac{d\beta}{dT}$$

$$= k_B \ln(2\cosh\beta J) - \frac{J}{T} \tanh\beta J$$

$$s = k_B\left[\ln(2\cosh\beta J) - \beta J \tanh\beta J\right]$$

At $T\to\infty$, $\beta\to 0$,

$$\cosh\beta J \simeq 1+\tfrac{1}{2}(\beta J)^2$$

$$\tanh(\beta J) \simeq \beta J$$

$$s \simeq k_B\left[\ln\left[2+(\beta J)^2\right] - (\beta J)^2\right]$$

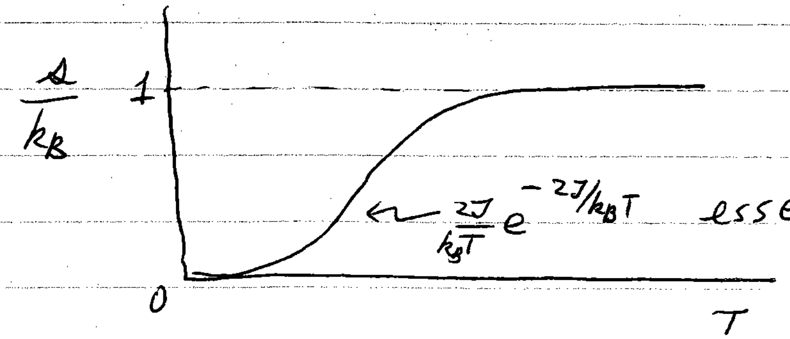
$$\simeq k_B \ln 2$$

At $T\to 0$, $\beta\to\infty$

$$\cosh\beta J \simeq e^{\beta J}$$

$$\tanh \simeq \frac{1-e^{-2\beta J}}{1+e^{-2\beta J}} \simeq 1 - 2e^{-2\beta J}$$

$$s \simeq k_B\left[\ln e^{\beta J} - \beta J(1-2e^{-2\beta J})\right] \simeq \frac{2J}{T} e^{-2J/k_B T}$$



$$C = T\left(\frac{\partial s}{\partial T}\right) = k_B T \left\{ \cancel{\frac{-2J\sinh\beta J}{2\cosh\beta J}\frac{1}{k_B T^2} + \frac{J}{k_B T^2}\tanh\beta J} \right.$$

$$\left. + \frac{\beta J^2}{k_B T^2}\frac{\partial}{\partial(\beta J)}\tanh\beta J \right\}$$

$$= \frac{J^2}{k_B T^2}\frac{\partial}{\partial(\beta J)}\left(\tanh\beta J\right) = \frac{J^2}{k_B T^2}\frac{1}{(\cosh\beta J)^2}$$

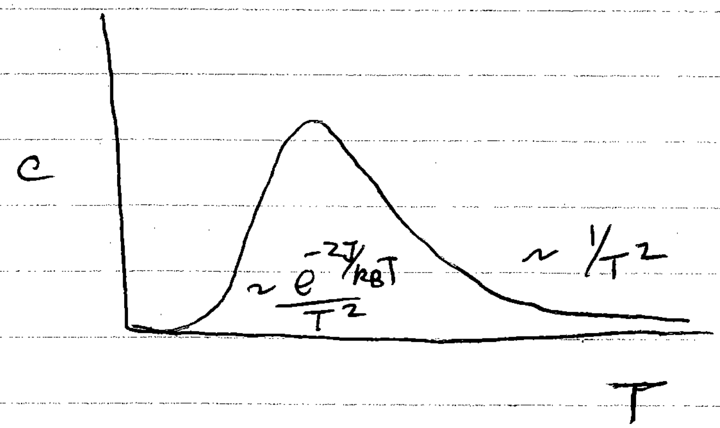
$$\boxed{C = k_B \left(\frac{\beta J}{\cosh\beta J}\right)^2}$$

as $T \to \infty$, $\beta \to 0$.

$$C \simeq k_B \left(\frac{J}{k_B T}\right)^2$$

as $T \to 0$, $\beta \to \infty$

$$C \simeq k_B \left(\frac{J}{k_B T}\right)^2 e^{-2J/k_B T}$$



essential singularity at $T=0$

$\Rightarrow$ No singularity at any finite T.

$\Rightarrow$ No phase transition at any finite T