

What behavior do we expect from Ising model?

For a given  $h$ , what is the resulting  $m(T, h)$ ?

For  $h > 0$ , expect  $m > 0$  as energetically favorable for spins to align parallel to  $h$ .

For  $h < 0$ , similarly expect  $m < 0$ .

In general,  $m(T, -h) = -m(T, h)$ , since Hamiltonian has the symmetry  $H[s_i, h] = H[-s_i, -h]$

What if  $h = 0$ ?

As  $T \rightarrow \infty$  we expect each spin to be random so  $m \rightarrow 0$ .

But even at finite  $T$  we might expect  $m = 0$  because of symmetry:  $H[s_i, 0] = H[-s_i, 0]$  so a configuration  $\{s_i\}$  in the partition function sum will enter with the same weight as the configuration  $\{-s_i\}$  and so expect  $\langle s_i \rangle = 0$ .

But at  $T = 0$ , the system has two degenerate ground states: all up or all down, with  $m = \pm 1$ . The ground state breaks the symmetry of the Hamiltonian.

More specifically:  $\lim_{h \rightarrow 0^+} \lim_{T \rightarrow 0} m(T, h) = +1$

limit  $h \rightarrow 0$  from above

limit  $h \rightarrow 0$  from below  $\rightarrow \lim_{h \rightarrow 0^-} \lim_{T \rightarrow 0} m(T, h) = -1$

Can one have such a broken symmetry state at finite  $T$ ?

$$\text{ie } \lim_{h \rightarrow 0^+} m(T, h) = m > 0$$

$$\lim_{h \rightarrow 0^-} m(T, h) = m < 0$$

For a finite size system,  $N$  finite, the answer is NO!

For a finite size system, the energy  $H[s_i]$  is always finite. The statistical weight of  $\{s_i\}$  will always be equal to that of  $\{-s_i\}$  in a small  $h$ , as we take  $h \rightarrow 0$

However, in the thermodynamic limit  $N \rightarrow \infty$ , the answer can be Yes! Now the energy of states with a finite  $\sum_i s_i$  will grow infinitely large as  $N$ ,

The statistical weight of config  $\{s_i\}$  can be infinitely different from that of  $\{-s_i\}$  in a small  $h$ , even if take  $h \rightarrow 0$ . ( $\infty \times 0 \neq 0$ )

$H[s_i] - H[-s_i] \propto hN$  does not necessarily vanish as  $h \rightarrow 0$ , if  $N \rightarrow \infty$  first

It is possible that at finite  $T$

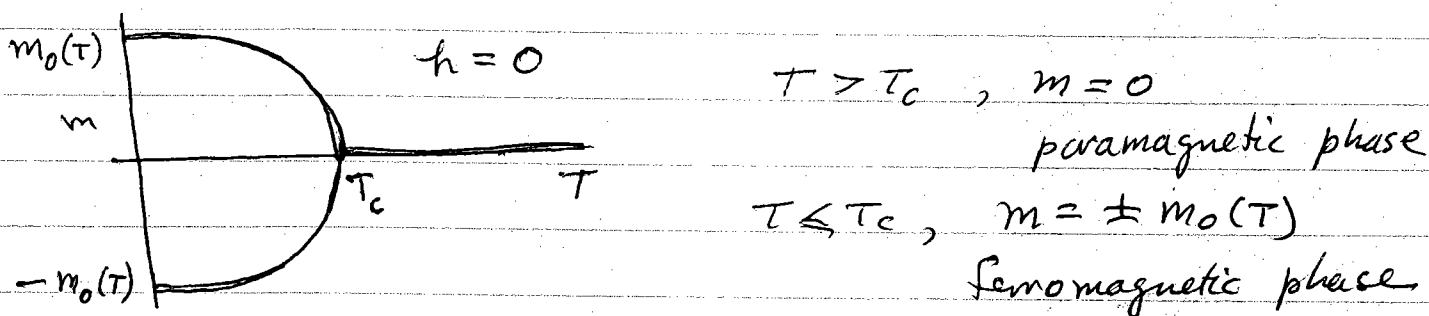
$$\lim_{h \rightarrow 0^+} \left[ \lim_{N \rightarrow \infty} m(T, h) \right] = m > 0$$

$$\lim_{h \rightarrow 0^-} \left[ \lim_{N \rightarrow \infty} m(T, h) \right] = m < 0$$

It is important to take the limits in the above order - ie first take  $N \rightarrow \infty$  in a finite  $h$ , and then take  $h \rightarrow 0$ . Reversing the limits ( $h \rightarrow 0$  first, then  $N \rightarrow \infty$ ) gives  $m=0$  by symmetry of  $H$ .

If such broken symmetry states exist at finite  $T$ , then do they persist at all  $T$ ? or do they disappear at a well defined  $T_c$ ?

Possibility of a phase transition

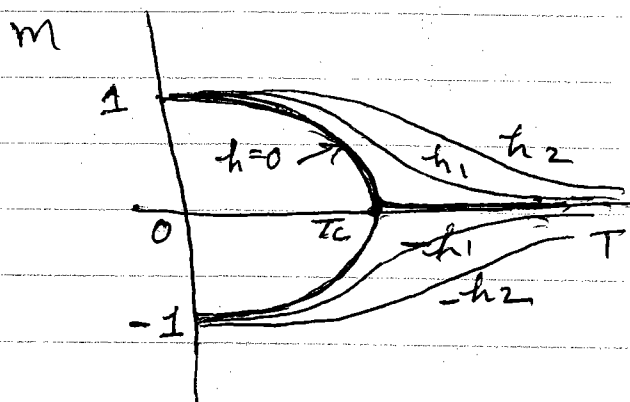


$m(T,0)$  is singular at  $T=T_c$

$T_c$  is ferromagnetic phase transition

The ordered state at  $T \leq T_c$  is a state of spontaneously broken symmetry. In  $h=0$  the system will pick either the up or the down state to order in, breaking the symmetry of the Hamiltonian.

At finite  $h$ , expect  $m(T, h)$  to behave like



$m(T, h)$  is smooth function of  $T$  for  $h \neq 0$ .

We said that to have a state of spontaneously broken symmetry at finite  $T$  ~~requires~~ one needs to be in thermodynamic limit  $N \rightarrow \infty$ .

Similarly, true singular phase transitions can only occur in the  $N \rightarrow \infty$  limit. Proof as follows:  
partition function sum:

$$Z(T, h) = \sum_{\{s_i\}} e^{-\beta H[s_i]}$$

For finite system ( $N$  finite) the number of configurations to sum over is  $2^N$  is finite.

$Z$  is therefore the sum of a finite number of analytic functions ("analytic" here in the sense of complex function theory - has no singularities as vary  $T, h$ ). As such,  $Z$  must itself be an analytic function of  $T$  at  $h$ .

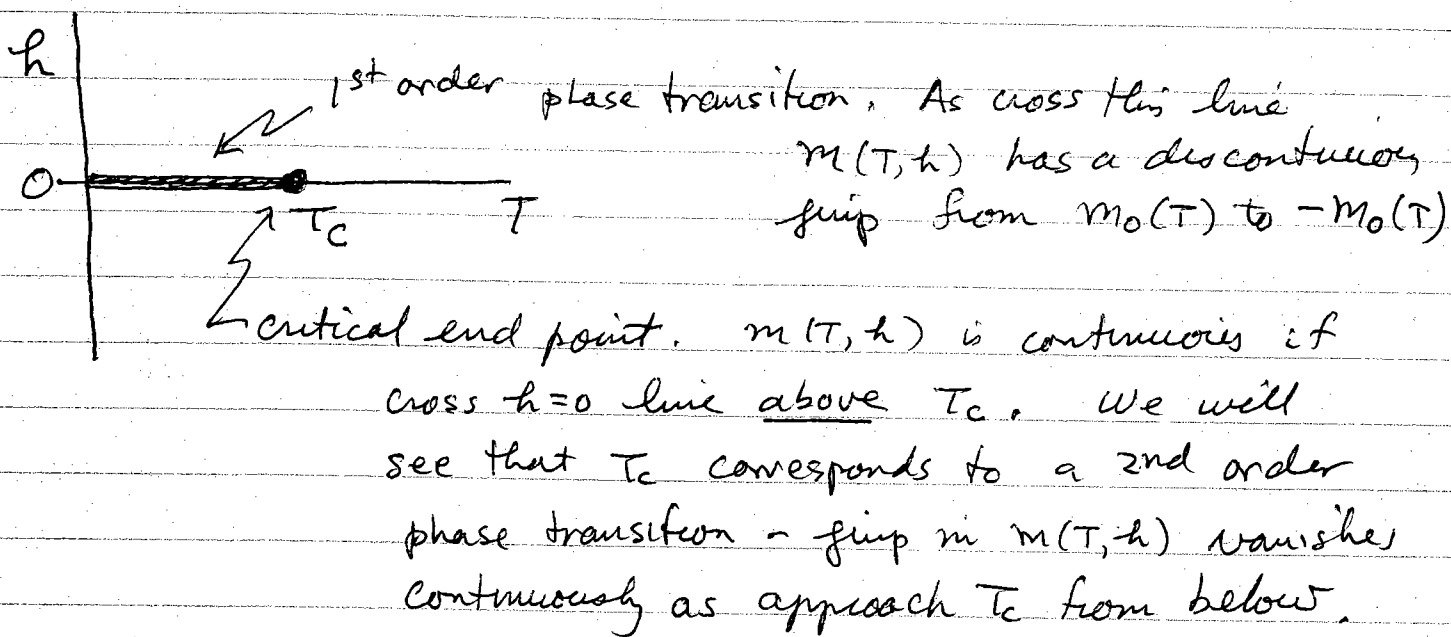
$\Rightarrow$   $Z$  can have no singularities

$\Rightarrow$  no singularities in any thermodynamic quantities

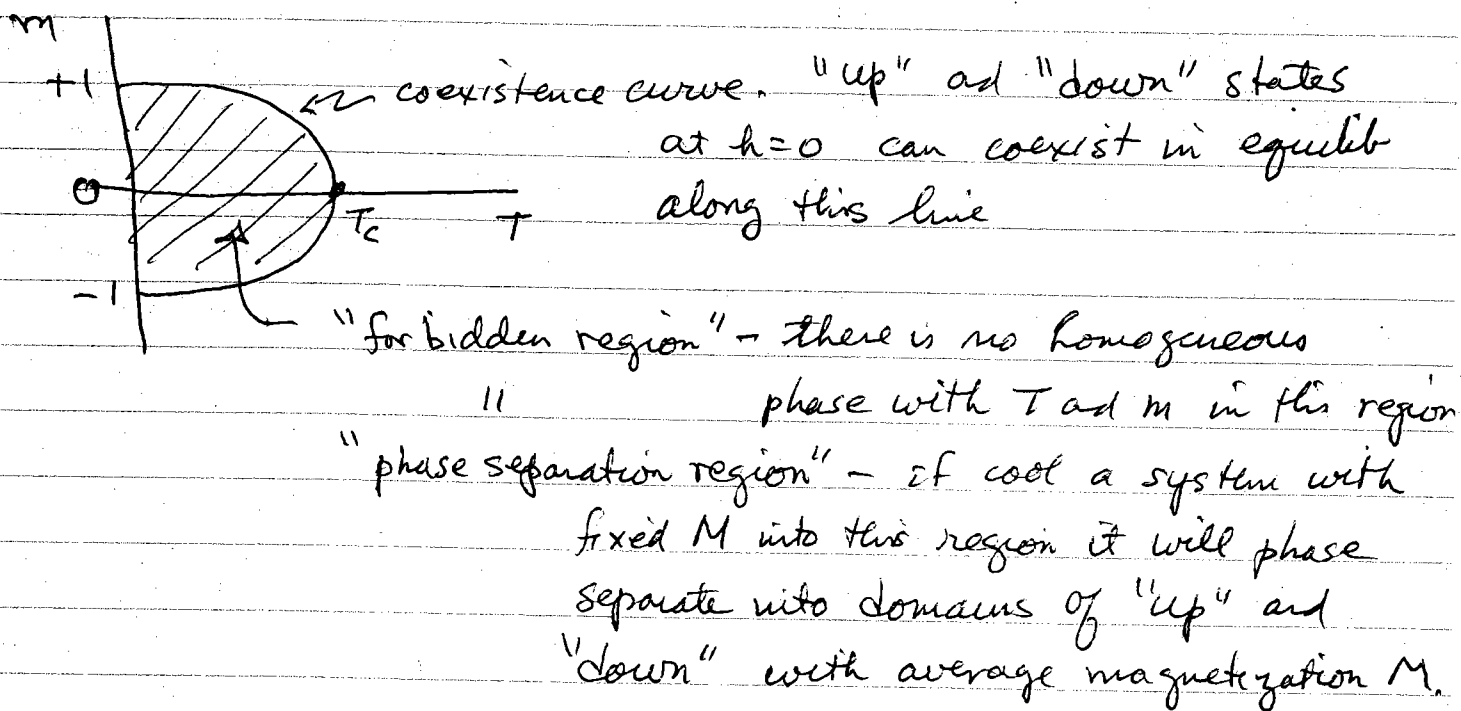
$\Rightarrow$  no phase transitions.

Only in thermodynamic limit of  $N \rightarrow \infty$  is  $Z$  now the sum of an infinite ~~and~~ number of analytic functions. Such an infinite sum need NOT be analytic, so phase transitions can exist.

## Phase diagram in $h-T$ plane



## Phase diagram in $m-T$ plane



Many similarities to liquid-gas phase diagram