

② At  $h=0$  on coexistence line

$$\left(1 - \frac{T_c}{T}\right)m + \left[\frac{1}{3}\left(\frac{T_c}{T}\right)^3 + \left(1 - \frac{T_c}{T}\right)\left(\frac{T_c}{T}\right)^2\right]m^3 = 0$$

as  $T \rightarrow T_c^-$ ,  $\left(1 - \frac{T_c}{T}\right) + \frac{1}{3}m^2 = 0$

$$m = \pm \sqrt{\frac{3(T_c - T)}{T}}$$

Define  $t = \frac{T_c - T}{T_c}$        $m \propto \pm \sqrt{3t} \propto t^\beta$        $\beta = 1/2$

③ At  $h=0$  on coexistence line as  $T \rightarrow T_c$

$$\frac{\partial h}{\partial m} = k_B T \left\{ \left(1 - \frac{T_c}{T}\right) + 3 \left[ \left(1 - \frac{T_c}{T}\right) \left(\frac{T_c}{T}\right)^2 + \frac{1}{3} \left(\frac{T_c}{T}\right)^3 \right] m^2 \right\}$$

$$\cong k_B T \left\{ \left(1 - \frac{T_c}{T}\right) + m^2 \right\}$$

As  $T \rightarrow T_c^+$  from above,  $m = 0$

$$\Rightarrow \frac{\partial h}{\partial m} = k_B T \left(1 - \frac{T_c}{T}\right) = k_B (T - T_c)$$

magnetic susceptibility  $\Rightarrow \frac{\partial m}{\partial h} = \chi^+ = \frac{1}{k_B (T - T_c)} \propto \frac{1}{|t|^\gamma}$        $\gamma = 1$

Note: at high temp  $T \gg T_c$ ,  $\chi \sim \frac{1}{T}$  just like in Curie paramagnetism. Hence we say the  $T > T_c$  phase is paramagnetic.

As  $T \rightarrow T_c^-$  from below,  $m^2 = 3 \left( \frac{T_c - T}{T} \right)$

$$\Rightarrow \frac{\partial h}{\partial m} = k_B T \left( \left(1 - \frac{T_c}{T}\right) + 3 \left(\frac{T_c - T}{T}\right) \right)$$

$$= 2k_B (T_c - T)$$

$$\frac{\partial m}{\partial h} = \chi^- = \frac{1}{2k_B (T_c - T)} \propto \frac{1}{|t|^\gamma} \quad \gamma = 1$$

also  $\lim_{T \rightarrow T_c} \left( \frac{\chi^+}{\chi^-} \right) = \frac{2k_B (T_c - T)}{k_B (T - T_c)} = 2 \leftarrow \text{amplitude ratio}$

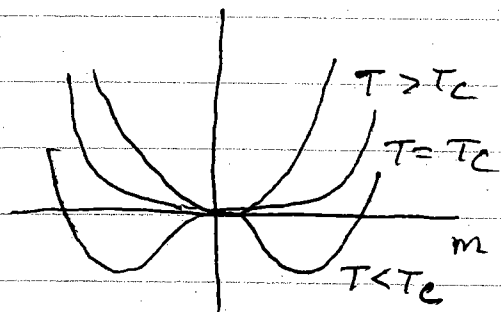
Our results here are identical to what we found for the van der Waal model of liquid-gas transition, if we make the identifications

$$L \leftrightarrow \delta p - \delta p^*(T) \quad \text{distance from coexistence curve}$$

$$m \leftrightarrow \delta v - \delta v_0$$

free energy  $f(m, T) - f(0, T) = \int_0^m h(m') dm'$

$$\Rightarrow f(m, T) - f(0, T) = k_B T \left\{ \frac{1}{2} \left(1 - \frac{T_c}{T}\right) m^2 + \frac{1}{12} m^4 \right\}$$

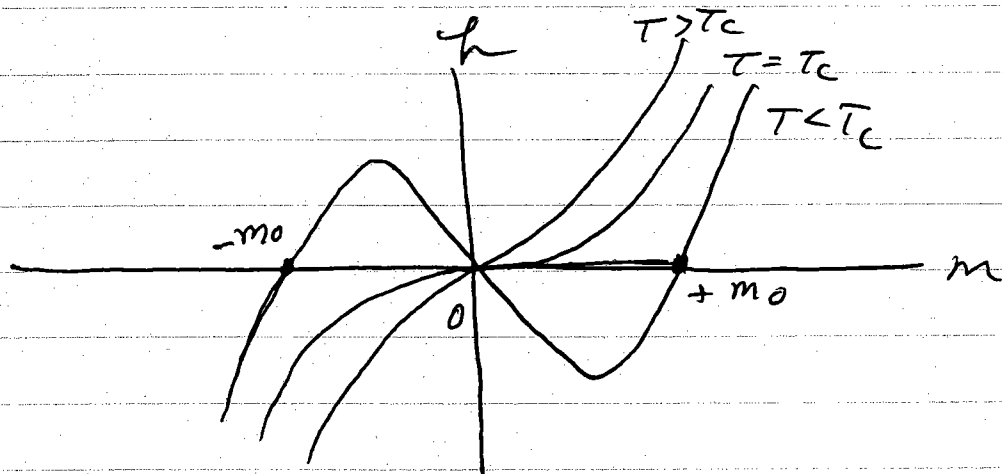


coefficient of  $m^2$  term vanishes at  $T_c$ , goes negative below  $T_c \Rightarrow$  minimum of  $f(m, T)$  changes from  $m=0$  to  $m = \pm m_0(T)$

$$g(h=0, T) = \min_m f(m, T) \Rightarrow \text{min of } f \text{ gives equilibrium state}$$

A closer look

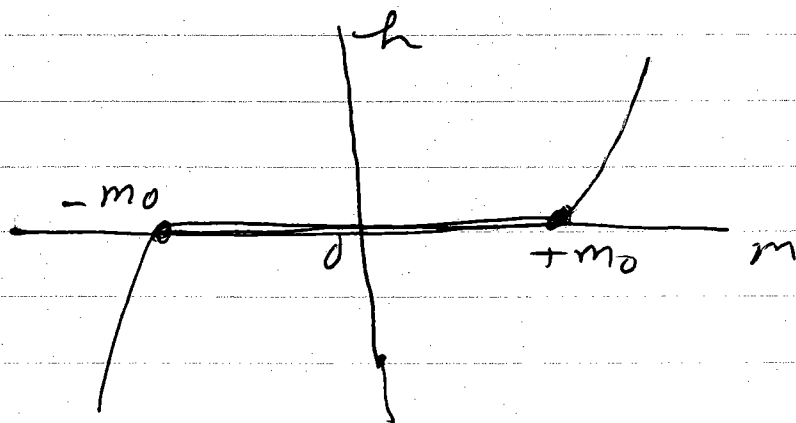
$$h = k_B T \left\{ \left(1 - \frac{T_c}{T}\right) m + \frac{1}{3} m^3 \right\}$$



For  $T < T_c$  we know that above  $h(m)$  curve cannot be valid for  $-m_0 \leq m \leq +m_0$ .

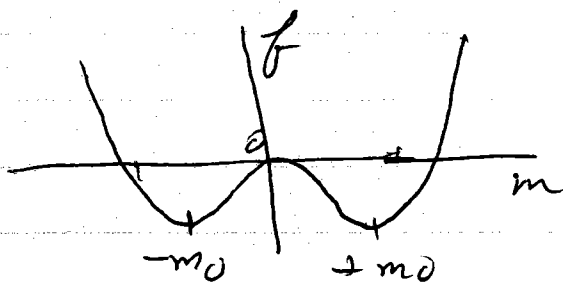
This is the coexistence region where  $h=0$

For  $T < T_c$ , the correct  $h(m)$  curve is

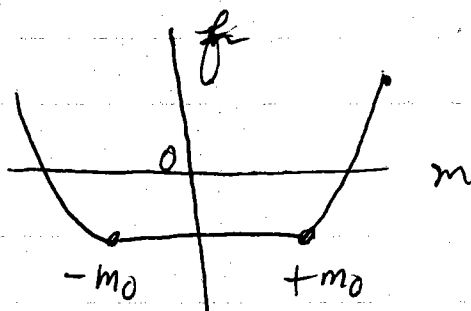


Such a "correction" based on our physical understanding is called the "Maxwell construction" originally done in connection with the van der Waals theory of the liquid to gas phase transition.

If we use the above  $h(m)$  for  $T < T_c$ , ~~then~~  
 to compute  $f(m, T)$ , then instead of



we get

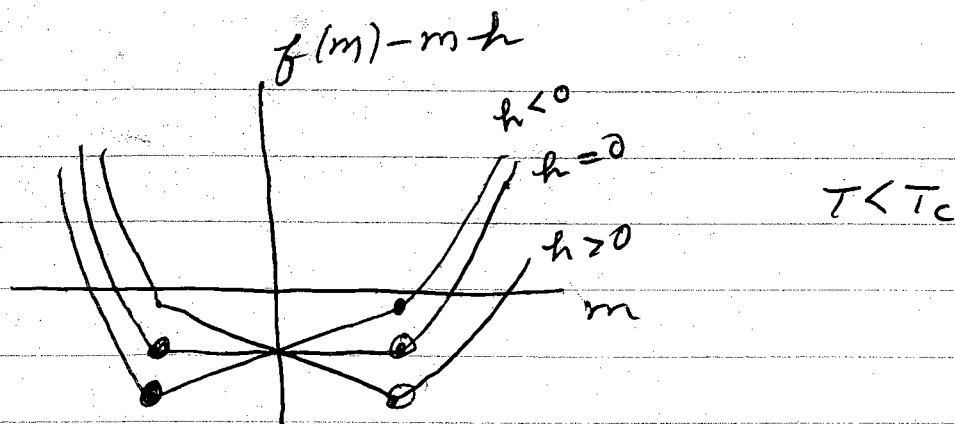


←  $f(m)$  with  
 Maxwell construction

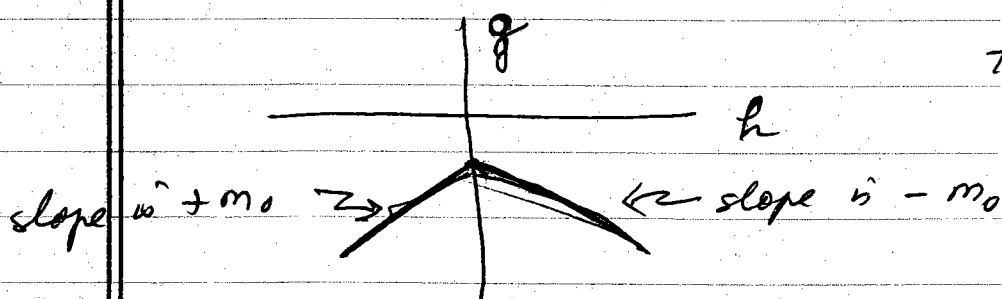
Note: This can be thought of as if we take the  
 top curve and replace it by its convex  
 envelop. The top curve cannot be physically  
 correct since  $f(m)$  must be convex in  $m$ .  
 Only the lower curve is convex.

Using the above corrected  $f(m)$ , we can  
 compute

$$g(h, T) = \min_m [f(m, T) - mh]$$



$g(h) = \min_m [f(m) - mh]$  then results in



$\frac{dg}{dh} = -m$  is discontinuous at  $h = 0$

$\Rightarrow g(h)$  has a cusp-like maximum at  $h = 0$

Note: The mean field approx is exact in the limit that every spin interacts with every other spin (not just nearest neighbors). Then

$$\begin{aligned} \mathcal{H} &= -\tilde{J} \sum_{i,j} s_i s_j - h \sum_i s_i \\ &= -\tilde{J} \sum_i s_i \left( \sum_j s_j \right) - h \sum_i s_i \\ &= -\tilde{J} \sum_i s_i N m - h \sum_i s_i \end{aligned}$$

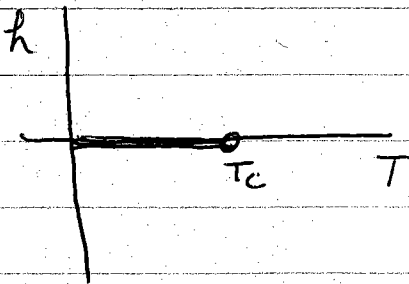
$$\mathcal{H} = -\left( \frac{z\tilde{J}}{2} m + h \right) \sum_i s_i$$

where we took  $J \equiv \frac{z\tilde{J}}{2}$ . In infinite range coupling model, need to take coupling  $J \propto \frac{1}{N}$  so that total energy scales with  $E \propto N$  as desired.

In the above,  $m[s_i] = \frac{1}{N} \sum_j s_j$  depends on the config  $\{s_i\}$ , however it is the same for every spin  $s_i$ .

# Landau's Theory of phase transitions

## Ising model



order parameter  $m = \begin{cases} \text{zero} & T > T_c \\ \text{non-zero} & T < T_c \end{cases}$   
ordering field  $h$

applying  $h \neq 0$  reduces symmetry of Hamiltonian, induces  $m \neq 0$ .

For ordering field  $h=0$ , Hamiltonian has higher symmetry. A finite order parameter  $m \neq 0$  breaks this symmetry.

For  $h=0$ , 2<sup>nd</sup> order phase transition at  $T_c$

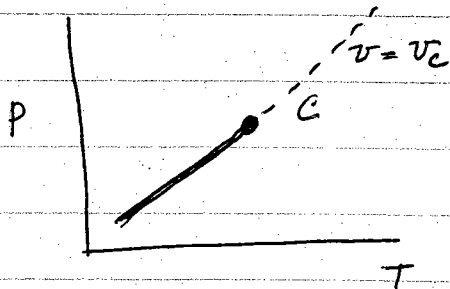
Such that  $m=0$  for  $T > T_c \Rightarrow$  thermodynamic state has full symmetry of  $\mathcal{H}$ . When  $T < T_c$

the order parameter becomes finite  $m \neq 0 \rightarrow$  thermodynamic state breaks symmetry of  $\mathcal{H}$ . Spontaneous symmetry breaking

For  $T < T_c$  varying the ordering field  $h$  through zero results in a discontinuous jump in the order parameter — 1<sup>st</sup> order transition line

~~At~~ the 2<sup>nd</sup> order transition at  $T_c$ , that ends the 1<sup>st</sup> order transition line,  $m$  goes to zero continuously as  $T \rightarrow T_c^-$ .

## For liquid-gas transition



1<sup>st</sup> order line does not have any particular symmetry with respect to the natural thermodynamic variables

dashed line is critical isochore - line of constant  $v = v_c$  smoothly extends from 1<sup>st</sup> order transition line.

Define ordering field  $\delta p$  as distance from critical isochore. Define order parameter  $\delta v = v - v_c$  as difference in specific volume (or density) from critical value.

## Landau methodology

- ① Given a physical system with a phase transition, first identify the order parameter  $m$  - a quantity that vanishes in the high  $T$  disordered phase, and is non zero in the low  $T$  ordered phase. Often this can be the hard part! cf. spin glass problem
- ② Near the 2<sup>nd</sup> order critical pt, the order parameter is small.  $\Rightarrow$  expand the Helmholtz free energy  $f(m, T)$  in a Taylor series in  $m$ , keeping all terms which have the appropriate symmetry of the problem.



For Ising model

$$f(m, T) \approx f_0 + am^2 + bm^4 + \dots$$

only even powers of  $m$  appear since  $H = -J \sum_{\langle ij \rangle} s_i s_j$   
is symmetric under  $\{s_i\} \rightarrow \{-s_i\}$ .

For liquid-gas transition

$$f(sv, T) \approx f_0 + asv^2 + bsv^3 + csv^4 + \dots$$

no symmetry of  $H$  to rule out odd powers of  
order parameter  $sv$ . - but still no linear  $sv$  term  
since  $f(sv=0, T)$  must be a minimum when  $T > T_c$   
- or equivalently, if there was a  $sv$  linear term in  
 $f(sv, T)$ , it would mean we were not properly  
expanding about critical isochore.

But we saw that by the trick of transformation  
of variables,  $sv = sv_0 + u$ , we can effectively  
eliminate the  $sv^3$  term and make the problem  
look just like the Ising model.

$$f(m, T) = f_0(T) + a(T)m^2 + b(T)m^4 + \dots$$

↑ ignore higher order terms

Stability  $\Rightarrow b(T) > 0$ ,  $f(m, T)$  must have global minimum

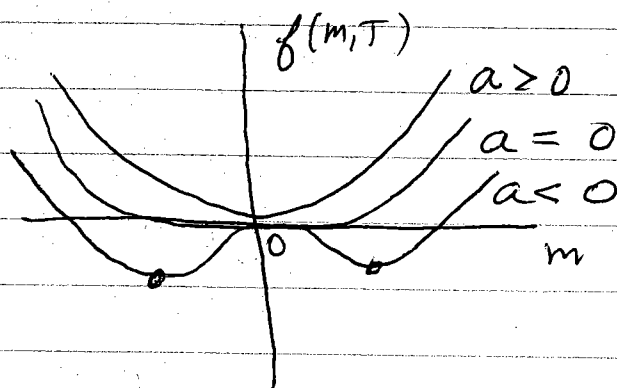
State of system is obtained by minimizing  $f(m, T)$  with respect to  $m$ . or equivalently, Gibbs free energy is

$$g(h, T) = \min_m [f(m, T) - hm]$$

on the 1<sup>st</sup> order line the ordering field  $h = 0$

$$\Rightarrow g(0, T) = \min_m [f(m, T)]$$

$\Rightarrow$  2<sup>nd</sup> order critical point occurs when  $a(T) = 0$



When  $a \geq 0$  then  $m = 0$  minimizes  $f(m, T)$   
 $\Rightarrow$  thermodynamic state has symmetry of  $H$

minimum of  $f(m, T)$  increases continuously from  $m = 0$  as  $a$  decreases below zero.

When  $a < 0$ , then  $m = \pm m_0$  minimizes  $f(m, T)$   
 $\Rightarrow$  thermodynamic state breaks symmetry