PHY 418 **Final Exam** **Spring 2011**

Answer all questions. Please put a box around your final answer for each part.

1) [20 points total] Consider a classical gas of N identical, non-relativistic, non-interacting atoms confined to a box of volume V in equilibrium at a temperature T. Each atom can be considered as a point particle, however it has an internal degree of freedom that can be in one of two possible states with energies $\epsilon_0 = 0$ and $\epsilon_1 > 0$ respectively.

a) [10 pts] Find the chemical potential μ of the gas as a function of T, V and N.

b) [10 pts] Find the specific heat per particle at constant volume, c_V, of the gas and sketch it as a function of temperature.

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2) [20 points total] Consider a classical gas of N identical, non-interacting, particles in the grand canonical ensemble with fugacity $z = \mathrm{e}^{\beta\mu}$, with μ the chemical potential and $\beta = 1/k_B T$.

a) [10 pts] Show that the probability $p(N)$ that there are exactly N particles in the system follows a Poisson distribution, i.e.,

$$p(N) = \frac{\lambda^N}{N!}\mathrm{e}^{-\lambda}$$

where $\lambda = \langle N \rangle$ is the average number of particles.

b) [10 pts] Find expressions for the average number of particles, $\langle N \rangle$, and for the fluctuation in the average number of particles, $\langle N^2 \rangle - \langle N \rangle^2$, in terms of the fugacity z and the single particle partition function.

Note: your answers to the above questions should apply for a particle with a general single-particle Hamiltonian, i.e. the particle could be more complicated than a simple free non-relativistic point particle.

(turn over for problems 3 and 4)

3) [30 points total] Consider a gas of N identical, non-relativistic, non-interacting spin 1/2 fermions (spin degeneracy $g_s = 2$), confined to move in a one-dimensional system of length L.

a) [10 pts] The density of states, $g(\epsilon)$, is defined as the number of single-particle states with energy ϵ per unit energy per unit length. Compute $g(\epsilon)$ for the gas.

b) [10 pts] Compute the Fermi energy of the gas as a function of its density $n = N/L$.

c) [10 pts] Compute the pressure of the gas at $T = 0$ as a function of its density n.

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4) [30 points total] Consider an ideal (i.e. non-interacting, identical particles) quantum gas in the grand canonical ensemble. The gas is in equilibrium at temperature T with chemical potential μ. In problem 3 of Problem Set 5, we showed that the number of particles n_i that occupy the single-particle energy eigenstate i is statistically independent of the number of particles n_j that occupy state j, and that the probability that state i will contain exactly n_i particles is given by,

$$p_i(n_i) = \frac{e^{-\beta(\epsilon_i - \mu)n_i}}{\sum_n e^{-\beta(\epsilon_i - \mu)n}} \ ,$$

where ϵ_i is the energy of single-particle energy eigenstate i, $\beta = 1/k_B T$, and the sum is over all allowed values of the integer occupancy number n. The above expression holds for both bosons and fermions, provided one uses the appropriate allowed values of n.

a) [10 pts] Using the above $p_i(n_i)$, derive the average occupation number of particles $\langle n_i \rangle$ in state i, for a gas of bosons and for a gas of fermions. (You must derive your results using the above formula, not just write down what you have memorized!)

b) [10 pts] Using the above $p_i(n_i)$, derive the occupation number fluctuation, $\langle n_i^2 \rangle - \langle n_i \rangle^2$, in state i, for a gas of bosons and for a gas of fermions.

c) [10 pts] If the total number of particles is $N = \sum_i n_i$, show that the fluctuation in N is given by,

$$\langle N^2 \rangle - \langle N \rangle^2 = \sum_i \left[\langle n_i^2 \rangle - \langle n_i \rangle^2 \right] \ .$$

Are the fluctuations in N for the quantum gas bigger or smaller than they are for the corresponding classical gas, for a gas of bosons? for a gas of fermions?