

Energy in the degenerate limit  $T=0$

$$\frac{\bar{E}}{V} = \int_0^{\epsilon_F} d\epsilon g(\epsilon) \epsilon$$

$$g(\epsilon) = C \sqrt{\epsilon}$$

$$\text{with } C = \left(\frac{2\pi m}{h^2}\right)^{3/2} \frac{2g_s}{\sqrt{\pi}}$$

$$n = \frac{N}{V} = \int_0^{\epsilon_F} d\epsilon g(\epsilon)$$

density of states

$$\Rightarrow \frac{\bar{E}}{V} = C \int_0^{\epsilon_F} d\epsilon \epsilon^{3/2} = \frac{2}{5} C \epsilon_F^{5/2}$$

$$n = \frac{N}{V} = C \int_0^{\epsilon_F} d\epsilon \epsilon^{1/2} = \frac{2}{3} C \epsilon_F^{3/2}$$

$$\Rightarrow \frac{\bar{E}}{V} = \frac{3}{5} \frac{N}{V} \epsilon_F$$

$$\frac{\bar{E}}{V} = \frac{3}{5} n \epsilon_F$$

or

$$\boxed{\frac{\bar{E}}{N} = \frac{3}{5} \epsilon_F}$$

↑ energy per volume

↑ energy per particle

Above gives  $T=0$  results. To get behavior at low  $T>0$ , or to get quantities such as  $C_V = \left(\frac{\partial E}{\partial T}\right)_V$ , we need to get the next order terms in a low temperature expansion.

In general we need to do integrals of the form

$$\int d\epsilon \frac{\tilde{\phi}(\epsilon)}{z^{-1} e^{\beta\epsilon} + 1} = \int d\epsilon \tilde{\phi}(\epsilon) n(\epsilon), \quad \tilde{\phi}(\epsilon) \text{ some function}$$

ex: to compute  $n$ ,  $\tilde{\phi}(\epsilon) = g(\epsilon)$ ; to compute  $\frac{\bar{E}}{V}$ ,  $\tilde{\phi}(\epsilon) = g(\epsilon) \epsilon$

transform variables to  $X = \beta \epsilon$ .

then we want to do integrals of the form

$$\Phi \equiv \int_0^{\infty} dx \frac{\phi(x)}{z^{-1} e^x + 1}$$

$\phi(x)$  is any function of  $x$ .

For example, to get the "standard" function  $f_n(z)$ , we use  $\phi(x) = \frac{1}{\Gamma(n)} x^{n-1}$

Define  $\xi = \beta \mu = \ln z$

$$\Phi = \int_0^{\infty} dx \frac{\phi(x)}{e^{x-\xi} + 1}$$

Define  $\psi(x) \equiv \int_0^x \phi(x') dx'$ ,  $f(x) \equiv \frac{1}{[e^{x-\xi} + 1]}$  fermi function

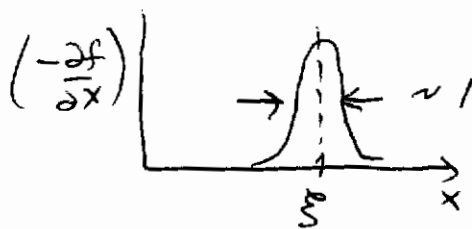
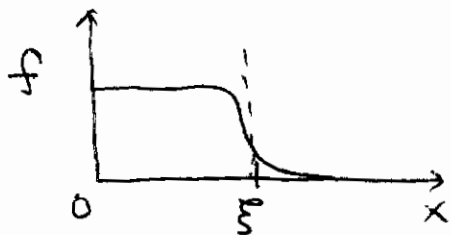
$$\Phi = \int_0^{\infty} dx \left( \frac{\partial \psi}{\partial x} \right) f(x) \quad \text{integrate by parts}$$

$$= \psi(x) f(x) \Big|_0^{\infty} + \int_0^{\infty} dx \psi(x) \left( -\frac{\partial f}{\partial x} \right)$$

$$= \int_0^{\infty} dx \psi(x) \left( -\frac{\partial f}{\partial x} \right) \quad \text{since } \psi(0) = 0 \text{ and } f(\infty) = 0$$

1st term vanishes

Now we use the fact that at low  $T$ ,  $\left( -\frac{\partial f}{\partial x} \right)$  is strongly peaked about  $x = \xi$



$\xi \gg 1$   
 $\xi \sim \frac{\epsilon_F}{kT}$  large

expand  $\psi(x)$  about  $x=\xi$

$$\psi(x) = \sum_{n=0}^{\infty} \left. \frac{d^n \psi}{dx^n} \right|_{x=\xi} \frac{(x-\xi)^n}{n!}$$

$$\Rightarrow \Phi = \sum_{n=0}^{\infty} \left. \frac{d^n \psi}{dx^n} \right|_{x=\xi} \int_0^{\infty} dx \frac{(x-\xi)^n}{n!} \left( -\frac{\partial f}{\partial x} \right)$$

since  $\left( -\frac{\partial f}{\partial x} \right)$  is zero except for a region of order 1 about  $x=\xi \gg 1$ , we can replace the lower limit of the integral by  $-\infty$  without any noticeable change

Then we can make a change of variables  $y = x - \xi$  and the integrals become

$$\int_{-\infty}^{\infty} dy \frac{y^n}{n!} \left( -\frac{\partial f}{\partial y} \right) \quad \text{where } f(y) = \frac{1}{e^y + 1}$$

$$\text{Now } -\frac{\partial f}{\partial y} = \frac{e^y}{(e^y + 1)^2} = \frac{e^y}{e^{2y} + 2e^y + 1} = \frac{1}{e^y + 2 + e^{-y}}$$

is symmetric about  $y=0$ .

$\Rightarrow$  all the integrals for  $n$  odd vanish!

To sum over only  $n$  even terms, let  $n \rightarrow 2n$

$$\Phi = \sum_{n=0}^{\infty} \frac{d^{2n} \phi}{dx^{2n}} \Big|_{x=\xi} \int_{-\infty}^{\infty} dy \frac{y^{2n}}{(2n)!} \left( -\frac{\partial f}{\partial y} \right)$$

$$\text{let } a_n \equiv \int_{-\infty}^{\infty} dy \frac{y^{2n}}{(2n)!} \left( -\frac{\partial f}{\partial y} \right), \quad a_0 = \int_{-\infty}^{\infty} dy \left( -\frac{\partial f}{\partial y} \right) = 1$$

The  $a_n$  are just numbers that be computed.  
They contain no system parameters whatsoever

For  $n \geq 1$  one can show

$$a_n = 2 \left( 1 - \frac{1}{2^{2n}} + \frac{1}{3^{2n}} - \frac{1}{4^{2n}} + \frac{1}{5^{2n}} - \dots \right)$$

$$= \left( 2 - \frac{1}{2^{2(n-1)}} \right) \zeta(2n)$$

where  $\zeta(n) = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \dots$  is the Riemann zeta function

$$\text{In particular } a_1 = \frac{\pi^2}{6}, \quad a_2 = \frac{7\pi^4}{360}$$

$$\Phi = \sum_{n=0}^{\infty} a_n \frac{d^{2n} \phi}{dx^{2n}} \Big|_{x=\xi} = \phi(\xi) + \sum_{n=1}^{\infty} a_n \frac{d^{2n} \phi}{dx^{2n}} \Big|_{x=\xi}$$

use  $\frac{d\phi}{dx} = \phi'$  to finally get

$$\phi(x) = \int_0^x dx' \phi(x')$$

$$\Phi = \int_0^{\xi} dx \phi(x) + \sum_{n=1}^{\infty} a_n \frac{d^{2n-1} \phi}{dx^{2n-1}} \Big|_{x=\xi}$$

$$= \int_0^{\xi} dx \phi(x) + \frac{\pi^2}{6} \frac{d\phi}{dx} \Big|_{x=\xi} + \frac{7\pi^4}{360} \frac{d^3 \phi}{dx^3} \Big|_{x=\xi} + \dots$$

This gives a power series in temperature.

To see this, transform back to the energy variable  
 $x = \beta \epsilon$ ,  $\epsilon = k_B T x$

$$\bar{\Phi} \equiv \int_0^{\infty} d\epsilon \frac{\phi(\epsilon)}{z^{-1} e^{\beta \epsilon} + 1} = k_B T \int_0^{\infty} dx \frac{\phi(k_B T x)}{z^{-1} e^x + 1}$$

using  $\int_0^{\mu/k_B T} dx \phi(k_B T x) = \int_0^{\mu} d\epsilon \phi(\epsilon)$

and  $\frac{d\phi}{dx} = \frac{d\phi}{d\epsilon} \frac{d\epsilon}{dx} = \frac{d\phi}{d\epsilon} k_B T$

we get

$$\bar{\Phi} = \int_0^{\infty} d\epsilon \phi(\epsilon) m(\epsilon)$$

$$\bar{\Phi} = \int_0^{\mu} d\epsilon \phi(\epsilon) + \frac{\pi^2 (k_B T)^2}{6} \left. \frac{d\phi}{d\epsilon} \right|_{\epsilon=\mu} + \frac{7\pi^4}{360} (k_B T)^4 \left. \frac{d^3 \phi}{d\epsilon^3} \right|_{\epsilon=\mu} + \dots$$

Example

① density  $n = \frac{N}{V} = \int_0^{\infty} d\epsilon g(\epsilon) m(\epsilon) \Rightarrow \phi(\epsilon) \equiv g(\epsilon)$

$$n = \int_0^{\mu} d\epsilon g(\epsilon) + \frac{\pi^2 (k_B T)^2}{6} \left. \frac{dg}{d\epsilon} \right|_{\epsilon=\mu} + \dots$$

Now as  $T \rightarrow 0$ ,  $\mu \rightarrow \epsilon_F$  the Fermi energy

$$n = \int_0^{\epsilon_F} d\epsilon g(\epsilon) + \int_{\epsilon_F}^{\mu} d\epsilon g(\epsilon) + \frac{\pi^2}{6} (k_B T)^2 \left. \frac{dg}{d\epsilon} \right|_{\epsilon=\mu}$$

But  $\epsilon_F$  was determined by  $n = \int_0^{\epsilon_F} d\epsilon g(\epsilon)$

$$\Rightarrow \int_{\epsilon_F}^{\mu} d\epsilon g(\epsilon) = -\frac{\pi^2}{6} (k_B T)^2 \left. \frac{dg}{d\epsilon} \right|_{\epsilon=\mu}$$

Since left hand side is  $o(kT)^2$  is small, we can approximate ~~the right hand side~~ as it as

$$\int_{\epsilon_F}^{\mu} d\epsilon g(\epsilon) \approx (\mu - \epsilon_F) g(\epsilon_F)$$

$$\Rightarrow (\mu - \epsilon_F) \approx -\frac{\pi^2}{6} \frac{(k_B T)^2}{g(\epsilon_F)} \left. \frac{dg}{d\epsilon} \right|_{\epsilon=\mu}$$

so  $\mu - \epsilon_F \sim o(k_B T)^2$  is small, so to lowest order can evaluate  $\frac{dg}{d\epsilon}$  on right hand side at  $\epsilon = \epsilon_F$  instead of  $\epsilon = \mu$

$$\boxed{\mu(T) \approx \epsilon_F - \frac{\pi^2}{6} (k_B T)^2 \frac{g'(\epsilon_F)}{g(\epsilon_F)}} \quad g' = \frac{dg}{d\epsilon}$$

Shows that chemical potential  $\mu$  decreases from  $\epsilon_F$  by  $o(kT)^2$  at low  $T$

For free electrons where  $g(\epsilon) = C\sqrt{\epsilon}$   
 $g'(\epsilon) = \frac{1}{2}C\frac{1}{\sqrt{\epsilon}}$

$$\mu(T) \approx \epsilon_F - \frac{\pi^2}{6} (k_B T)^2 \frac{1}{2\epsilon_F} = \epsilon_F - \frac{\pi^2}{12} \frac{(k_B T)^2}{\epsilon_F}$$

$$\mu(T) \approx \epsilon_F \left( 1 - \frac{1}{3} \left( \frac{\pi k_B T}{2\epsilon_F} \right)^2 \right) = \epsilon_F \left( 1 - \frac{1}{3} \left( \frac{\pi T}{2 T_F} \right)^2 \right)$$

Correction is small for metals at room temp as  $T_F \sim 10,000^\circ\text{K}$

② energy  $\frac{E}{V} = \int_0^\infty d\epsilon g(\epsilon) \epsilon n(\epsilon) \Rightarrow \phi(\epsilon) = g(\epsilon) \epsilon$

$$u = \frac{E}{V} = \int_0^\mu d\epsilon g(\epsilon) \epsilon + \frac{\pi^2}{6} (k_B T)^2 [g(\mu) + \mu g'(\mu)]$$

$$= \underbrace{\int_0^{\epsilon_F} d\epsilon g(\epsilon) \epsilon}_{= u(0)} + \underbrace{\int_{\epsilon_F}^\mu d\epsilon g(\epsilon) \epsilon}_{\approx (\mu - \epsilon_F) g(\epsilon_F) \epsilon_F} + \frac{\pi^2}{6} (k_B T)^2 [g(\mu) + \mu g'(\mu)]$$

ground state energy density
as before
replace  $\mu \approx \epsilon_F$  as before

$$u(T) = u(0) + (\mu - \epsilon_F) g(\epsilon_F) \epsilon_F + \frac{\pi^2}{6} (k_B T)^2 [g(\epsilon_F) + \epsilon_F g'(\epsilon_F)]$$

$$= u(0) + \left[ -\frac{\pi^2}{6} (k_B T)^2 \frac{g'(\epsilon_F)}{g(\epsilon_F)} \right] g(\epsilon_F) \epsilon_F + \frac{\pi^2}{6} (k_B T)^2 [g(\epsilon_F) + \epsilon_F g'(\epsilon_F)]$$

$$u(T) = u(0) + \frac{\pi^2}{6} (k_B T)^2 g(\epsilon_F)$$

specific heat per volume

$$C_V \equiv \frac{C_V}{V} = \frac{1}{V} \left( \frac{dE}{dT} \right) = \left( \frac{dU}{dT} \right) / V$$

$$C_V = \frac{\pi^2}{3} k_B^2 T g(E_F)$$

for free electrons we can write  $g(E) = C\sqrt{E}$

$$m = \int_0^{E_F} dE g(E) = \frac{2}{3} C E_F^{3/2} \Rightarrow C = \frac{3}{2} \frac{m}{E_F^{3/2}}$$

$$\Rightarrow g(E_F) = \frac{3}{2} \frac{m}{E_F^{3/2}} \cdot E_F^{1/2} = \frac{3}{2} \frac{m}{E_F} \quad \text{density of states at fermi energy}$$

$$C_V = \frac{\pi^2}{2} \left( \frac{k_B T}{E_F} \right) m k_B$$

or total specific heat  $C_V = V c_V$   $mV = N$

$$C_V = \frac{\pi^2}{2} \left( \frac{k_B T}{E_F} \right) N k_B$$

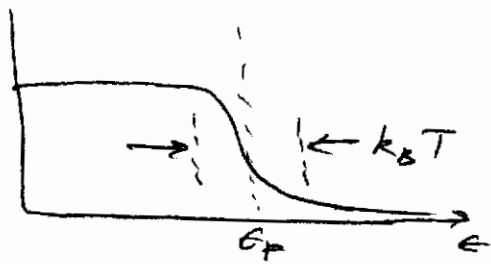
$\Rightarrow$  specific heat due to fermi gas of electrons in a conductor is  $C_V \sim T$  at low temperatures

We already saw that specific heat due to ionic vibrations (phonons) in a solid went like  $C_V \sim T^3$  at low temperatures (Debye model)

$\Rightarrow$  electronic contribution to  $C_V$  dominates at sufficiently low  $T$ .



## Simple estimate of $C_V$



When increase temperature to  $k_B T$ , the electrons near the Fermi energy  $\epsilon_F$  will increase their energy by an amount  $\sim k_B T$ . The number of such electrons ~~is roughly~~ per unit volume is roughly

$$g(\epsilon_F)(k_B T)$$

↑ density of states at  $\epsilon_F$   
 ↑ energy interval about  $\epsilon_F$  of states which ~~increase~~ get excited

⇒ increase in energy per unit volume is

$$\Delta U \sim (g(\epsilon_F) k_B T) (k_B T) \sim g(\epsilon_F) (k_B T)^2$$

↑  
# electrons excited

↑  
excitation energy per excited electron

for free electrons

$$\Rightarrow C_V = \frac{d\Delta U}{dT} \sim g(\epsilon_F) k_B^2 T = \frac{3}{2} \frac{m}{\epsilon_F} k_B^2 T = \frac{3}{2} m k_B \frac{T}{\epsilon_F}$$

The previous calculation gives the precise numerical coefficient

electronic specific heat per volume

$$C_V^{\text{elec}} = \frac{\pi^2}{2} \left( \frac{k_B T}{E_F} \right) \frac{N k_B}{V} \left( 1 + o \left( \frac{k_B T}{E_F} \right)^2 \right)$$

compare to classical result  $C_V^{\text{classical}} = \frac{N k_B}{V}$

The correct result for degenerate fermi gas is a factor

$$\frac{\pi^2}{2} \left( \frac{k_B T}{E_F} \right) = \frac{\pi^2}{2} \left( \frac{T}{T_F} \right) \text{ smaller than classical result by factor } \sim \frac{10^2}{10^4} = 10^{-2} \text{ at room temperature}$$

also, classical  $C_V$  is indep of  $T$ , whereas fermi gas result is  $\propto T$ .

At low  $T$ , the ionic contribution to  $C_V$  is

$$C_V^{\text{ion}} = \frac{12\pi^4}{5} \left( \frac{T}{\Theta_D} \right)^3 \frac{N k_B}{V}$$

$$\frac{C_V^{\text{elec}}}{C_V^{\text{ion}}} = \frac{\pi^2}{2} \left( \frac{k_B T}{E_F} \right) \frac{5}{12\pi^4} \left( \frac{\Theta_D}{T} \right)^3 \approx \frac{5}{24\pi^2} \left( \frac{\Theta_D}{T_F} \right) \left( \frac{\Theta_D}{T} \right)^2$$
$$\approx 1 \text{ when } T^* = \sqrt{\frac{5}{24\pi^2} \left( \frac{\Theta_D}{T_F} \right)} \Theta_D \approx 0.15 \left( \frac{\Theta_D}{T_F} \right)^{1/2} \Theta_D$$

for metals,  $T_F \sim 10^4 \text{ } ^\circ\text{K}$ ,  $\Theta_D \sim 10^2 \text{ } ^\circ\text{K}$

$$T^* = 0.15 \sqrt{10^{-2}} \Theta_D \approx 0.015 \Theta_D$$

So ionic contrib to  $C_V$  dominates over electronic contrib until  $T \lesssim 0.01 \Theta_D$  i.e. at  $0(1) \text{ } ^\circ\text{K}$ . The electronic contrib dominates at lower temperatures.