we can work in fixed magnetization or fixed magnetic field ensemble according to our convenience. Usually it is easiest to work with fixed magnetic field. In this case we usually write

$$\mathcal{H} = -J \sum_{\langle ij \rangle} s_i s_j - h \sum_i s_i$$

including the magnetic field part in the definition of $\mathcal{H}$.

$$Z = \sum_{\{s_i\}} e^{-\beta \mathcal{H}} \quad \text{includes } h \text{ term}$$

define magnetization density

$$m = \frac{M}{N} = \frac{1}{N} \langle \sum_i s_i \rangle \qquad N = \text{total number spins}$$

Helmholtz free energy density: In limit $N \to \infty$, $F(T,M) = N f(T,m)$

$\frac{F}{N} \equiv f(T,m)$ depends on magnetization density

$$df = -s\, dT + h\, dm \qquad s = \frac{S}{N} \text{ entropy per spin}$$

Gibbs free energy density: In limit $N \to \infty$, $G(T,h) = N g(T,h)$

$\frac{G}{N} \equiv g(T,h)$

$$dg = -s\,dT - m\, dh$$

$$\left(\frac{\partial f}{\partial m}\right)_T = h \quad , \quad \left(\frac{\partial g}{\partial h}\right)_T = -m$$

What behavior do we expect from Ising model?
For a given h, what is the resulting $m(T,h)$?

For $h>0$, expect $m>0$ as energetically favorable for spins to allign parallel to h.
For $h<0$, similarly expect $m<0$.

In general, $m(T,-h) = -m(T,h)$, since Hamiltonian has the symmetry $\mathcal{H}[s_i,h] = \mathcal{H}[-s_i,-h]$

<u>What if $h=0$?</u>

As $T\to\infty$ we expect each spin to be random so $m\to 0$. But even at finite T we might expect $m=0$ because of symmetry: $\mathcal{H}[s_i,0] = \mathcal{H}[-s_i,0]$ so a configuration $\{s_i\}$ in the partition function sum will enter with the same weight as the configuration $\{-s_i\}$ and so expect $\langle s_i\rangle = 0$.

But at $T=0$, the system has two <u>degenerate</u> ground states: <u>all up</u> or <u>all down</u>, with $m=\pm 1$. The ground state <u>breaks</u> the symmetry of the Hamiltonian.

<u>More specifically</u>: $\lim\limits_{h\to 0^+}\lim\limits_{T\to 0} m(T,h) = +1$

limit $h\to 0$ from above

limit $h\to 0$ from below $\lim\limits_{h\to 0^-}\lim\limits_{T\to 0} m(T,h) = -1$

Can one have such a broken symmetry state at finite T?

i.e. $\lim_{h \to 0^+} m(T, h) = m > 0$

$\lim_{h \to 0^-} m(T, h) = m < 0$

For a <u>finite size system</u>, N finite, the answer is <u>NO</u>! For a finite size system, the energy $\mathcal{H}[s_i]$ is always finite. The statistical weight of $\{s_i\}$ will always be equal to that of $\{-s_i\}$ in a small h, as we take $h \to 0$

However, in the <u>thermodynamic limit</u> $N \to \infty$, the answer can be <u>Yes</u>! Now the energy of states with a finite $\sum_i s_i$ will grow infinitely large as N, The statistical weight of config $\{s_i\}$ can be infinitely different from that of $\{-s_i\}$ in a small h, even if take $h \to 0$. $(\infty \times 0 \neq 0)$
$\mathcal{H}[s_i] - \mathcal{H}[-s_i] \propto hN$ does not necessarily vanish as $h \to 0$ if $N \to \infty$ first.
It is possible that at finite T

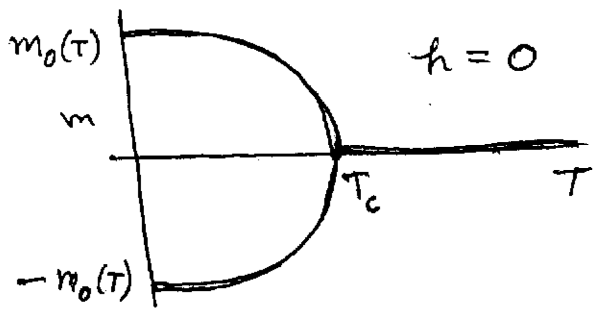
$$\lim_{h \to 0^+} \left[\lim_{N \to \infty} m(T, h) \right] = m > 0$$

$$\lim_{h \to 0^-} \left[\lim_{N \to \infty} m(T, h) \right] = m < 0$$

It is important to take the limits in the above order – i.e. first take $N \to \infty$ in a finite h, and then take $h \to 0$. Reversing the limits ($h \to 0$ first, then $N \to \infty$) gives $m = 0$ by symmetry of $\mathcal{H}$.

If such broken symmetry states exist at finite T, then do they persist at all T? or do they disappear at a well defined T_c?

Possibility of a phase transition



$T > T_c$, $m = 0$
paramagnetic phase

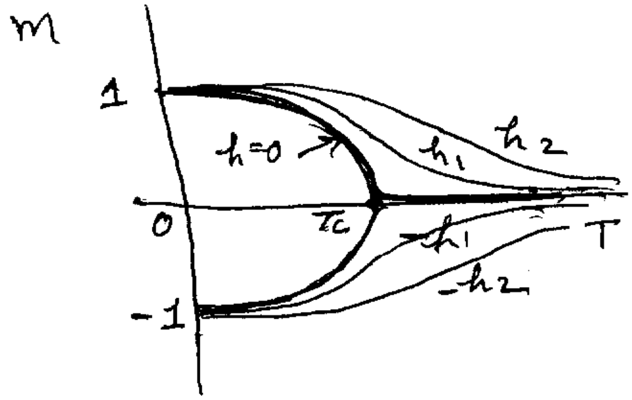
$T \leq T_c$, $m = \pm m_0(T)$
ferromagnetic phase

$m(T, 0)$ is singular at $T = T_c$

T_c is ferromagnetic phase transition

The ordered state at $T \leq T_c$ is a state of spontaneously broken symmetry. In $h = 0$ the system will pick either the up or the down state to order in, breaking the symmetry of the Hamiltonian.

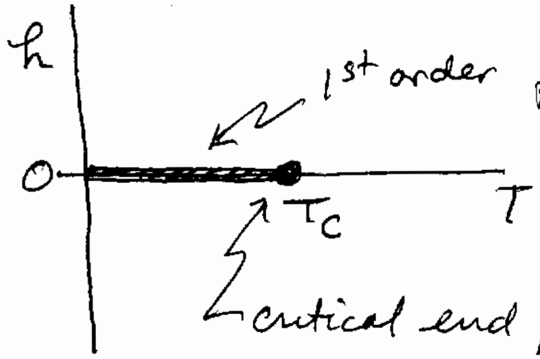
At finite h, expect $m(T, h)$ to behave like



$h_1 < h_2$

$m(T, h)$ is smooth function of T for $h \neq 0$.

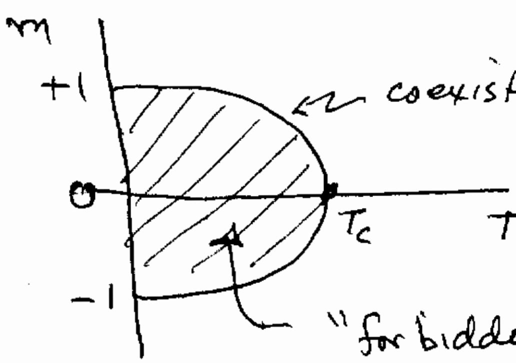
Phase diagram in $h-T$ plane



1st order phase transition. As cross this line $m(T,h)$ has a discontinuous jump from $m_0(T)$ to $-m_0(T)$

critical end point. $m(T,h)$ is continuous if cross $h=0$ line above T_c. We will see that T_c corresponds to a 2nd order phase transition - jump in $m(T,h)$ vanishes continuously as approach T_c from below.

Phase diagram in $m-T$ plane



coexistence curve. "up" and "down" states at $h=0$ can coexist in equilib along this line

"forbidden region" - there is no homogeneous phase with T and m in this region

" "

"phase separation region" - if cool a system with fixed M into this region it will phase separate into domains of "up" and "down" with average magnetization M.

Many similarities to liquid-gas phase diagram

We said that to have a state of spontaneously broken symmetry at finite T ~~[illegible]~~ one needs to be in thermodynamic limit $N \to \infty$.

Similarly, true singular phase transitions can only occur in this $N \to \infty$ limit. Proof as follows:
partition function sum:

$$Z(T, h) = \sum_{\{S_i\}} e^{-\beta H[S_i]}$$

For finite system (N finite) the number of configurations to sum over is 2^N is finite.

Z is therefore the sum of a finite number of analytic functions ("analytic" here in the sense of complex function theory - has no singularities as vary T, h). As such, Z must itself be an analytic function of T and h.
$\Rightarrow$ Z can have no singularities
$\Rightarrow$ no singularities in any thermodynamic quantities
$\Rightarrow$ no phase transitions.

Only in thermodynamic limit of $N \to \infty$ is Z now the sum of an infinite ~~[illegible]~~ number of analytic functions. Such an infinite sum need NOT be analytic, so phase transitions can exist.