

Back to Ising model : Take  $\phi(m, T, h)$   
 $= f(m, T) - hm$

$$f(m, T) = f_0(T) + a(T)m^2 + b(T)m^4 + \dots$$

↑ ignore higher order terms

Stability  $\Rightarrow b(T) > 0$  ;  $f(m, T)$  must have global minimum

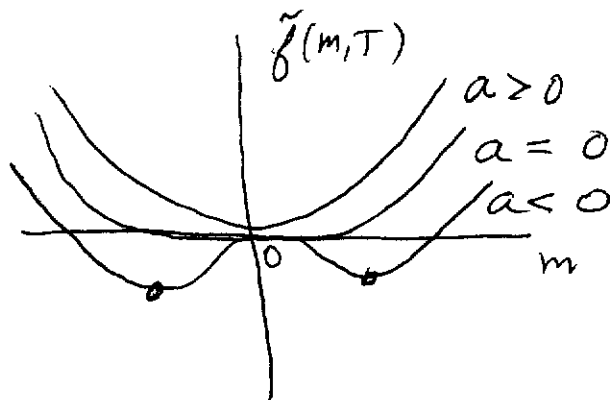
State of system is obtained by minimizing  $f(m, T)$  with respect to  $m$ . Or equivalently, Gibbs free energy is

$$g(h, T) = \min_m [f(m, T) - hm]$$

On the 1<sup>st</sup> order line the ordering field  $h = 0$

$$\Rightarrow g(0, T) = \min_m [f(m, T)]$$

$\Rightarrow$  2<sup>nd</sup> order critical point occurs when  $a(T) = 0$



minimum of  $f(m, T)$  increases  
continuously from  $m=0$   
 as  $a$  decreases below zero.

When  $a > 0$  then  
 $m=0$  minimizes  $f(m, T)$   
 $\Rightarrow$  thermodynamic state  
 has symmetry of  $H$

When  $a < 0$ , then  $m = \pm m_0$   
 minimizes  $f(m, T)$   
 $\Rightarrow$  thermodynamic state  
 breaks symmetry

Expanding near  $T_c$  to lowest orders,

$$b(T) \approx b(T_c) \equiv b_0 \quad \text{a constant}$$

$$a(T) \approx a_0 [T - T_c] \quad a_0 \text{ a constant}$$

① Behaviour of order parameter near  $T_c$

$T < T_c$  minimise  $\beta(m, T)$

$$\Rightarrow 2a m + 4b m^3 = 0$$

$$2a + 4b m^2 = 0 \quad \text{for } m \neq 0$$

$$m^2 = -\frac{a}{2b}$$

$$\boxed{m_0 = \pm \sqrt{\frac{a_0 (T_c - T)}{2b_0}}} \propto |t|^\beta \quad \boxed{\beta = 1/2}$$

$$t = \left( \frac{T_c - T}{T_c} \right)$$

same  $\beta$  as found earlier

②  $h(m)$  curve at critical isotherm  $T = T_c$

$$g(h, T) = \min_m [f(m, T) - h m]$$

$$= \min_m [f_0 + b_0 m^4 - h m] \quad a=0 \text{ at } T_c$$

$$\Rightarrow 4b_0 m^3 - h = 0 \quad \Rightarrow \boxed{h = 4b_0 m^3}$$

$$h \propto m^\delta \quad \boxed{\delta = 3} \quad \text{same as before}$$

③ susceptibility  $\chi = \frac{\partial m}{\partial h}$  at  $h=0$

$$g(h, T) = \min_m [f(m, T) - hm]$$

$$\Rightarrow 2am + 4bm^3 = h \quad \text{"equation of state"}$$

$$\chi^{-1} = \frac{\partial h}{\partial m} = 2a + 12bm^2$$

$$\chi = \frac{1}{2a + 12bm^2}$$

For  $T > T_c$ ,  $h=0 \Rightarrow m^2=0$

$$\boxed{\chi^+ = \frac{1}{2a}} = \frac{1}{2a_0(T-T_c)} \propto \frac{1}{|t|} \gamma, \quad \boxed{\gamma = 1}$$

For  $T < T_c$ ,  $h=0 \Rightarrow m^2 = m_0^2 = \frac{-a}{2b} = \frac{a_0(T_c - T)}{2b_0}$

$$\chi^- = \frac{1}{2a_0(T-T_c) + \frac{12b_0 a_0 (T_c - T)}{2b_0}}$$

$$\boxed{\chi^- = \frac{1}{4a_0(T_c - T)}} \propto \frac{1}{|t|} \gamma \quad \boxed{\gamma = 1}$$

$$\boxed{\lim_{T \rightarrow T_c} \frac{\chi^+}{\chi^-} = 2}$$

amplitude ratios

all same as before

④ specific heat at  $h=0$  along 1<sup>st</sup> order transition line

from ① we have  $m_0^2 = -\frac{a}{2b}$   $T < T_c$ ,  $m_0^2 = 0$   $T > T_c$

$$\Rightarrow g(h=0, T) = f(m_0, T) = f_0(T), T > T_c$$

$$= f_0(T) + a\left(\frac{-a}{2b}\right) + b\left(\frac{-a}{2b}\right)^2, T < T_c$$

$$T < T_c: f(m_0, T) = f_0(T) - \frac{a^2}{2b} + \frac{a^2}{4b} = f_0(T) - \frac{a^2}{4b}$$

$$= f_0(T) - \frac{a_0^2}{4b_0} (T - T_c)^2$$

specific heat

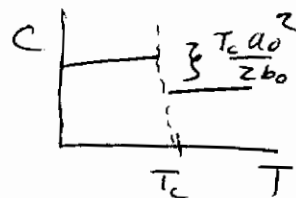
$$\Delta = -\frac{\partial g}{\partial T} \Rightarrow C = T \left( \frac{\partial \Delta}{\partial T} \right)_{h=0} = -T \frac{\partial^2 g}{\partial T^2}$$

$$C = -T \frac{d^2 f(m_0(T), T)}{dT^2}$$

$$= \begin{cases} -T \frac{d^2 f_0}{dT^2} & T > T_c \end{cases}$$

$$\begin{cases} -T \frac{d^2 f_0}{dT^2} + \frac{T a_0^2}{2b_0} & T < T_c \end{cases}$$

$$\Rightarrow C(T \rightarrow T_c^-) - C(T \rightarrow T_c^+) = \frac{T_c a_0^2}{2b_0}$$



jump in specific heat at  $T_c$

The piece  $\frac{\partial^2 f_0}{\partial T^2}$  is the non singular piece of the specific heat.  $f_0$  is the same as the "reference" free energy we used earlier when integrating the equation of state in the mean field or the van der Waals approx.

We can define a critical exponent  $\alpha$  for the specific heat by  $C \propto |t|^{-\alpha}$ , or

$$\alpha = \lim_{t \rightarrow 0} \left[ \frac{\ln C}{-\ln |t|} \right]$$

For Landau theory this gives  $\boxed{\alpha = 0}$

Summary: Landau theory = mean field theory

$$h=0, \quad m_0(T) \sim |t|^{\beta} \quad \underline{\beta = 1/2}$$

$$T=T_c, \quad h(m) \propto m^{\delta} \quad \underline{\delta = 3}$$

$$h=0, \quad \chi(T) \propto \frac{1}{|t|^{\gamma}} \quad \underline{\gamma = 1}$$

$$\lim_{t \rightarrow 0} \frac{\chi^+}{\chi^-} = 2$$

$$h=0, \quad C(T) \propto |t|^{-\alpha} \quad \underline{\alpha = 0}$$

} mean field critical exponents

exponent values in mean field approx are indep of dimension  $d$ .

From exact solution of 2D Ising model

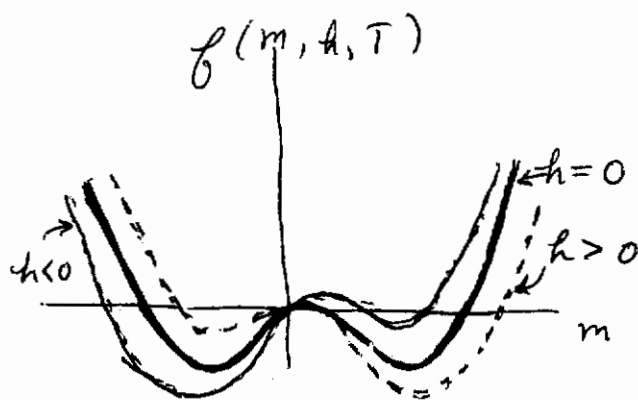
$$\delta = 15 \quad \beta = 1/8, \quad \gamma = 7/4, \quad \alpha = 0, \quad \nu = 1 \quad C \propto \ln |t|$$

log divergence

## Landau theory of 1st order transition

For  $T < T_c$ ,  $h \neq 0$

$$g(h, T) = \min_m [f(m, T) - hm] \equiv \min_m [f(m, h, T)]$$



as  $h$  goes smoothly through zero, the value of  $m$  that minimizes  $f(m, h, T)$  jumps discontinuously from  $+m_0$  to  $-m_0$ .

2nd order transition - order parameter goes continuously to zero  
1st order transition - order parameter jumps discontinuously

Note: Landau theory  $\equiv$  mean field theory  
 gives the same values of the critical exponents  
independent of dimension  $d$ , and number of  
 components of spin  $n$ .

For  $n$ -component spins with  $\vec{m} = \frac{1}{N} \sum_i \vec{s}_i$

$$f(\vec{m}, T) = f_0 + a|\vec{m}|^2 + b|\vec{m}|^4 + \dots$$

everything comes out the same!

But can get some interesting new behaviors by doing other things.

①  $f(m, T) = f_0 + a m^2 - b m^4 + c m^6$

$b > 0 \Rightarrow$  quartic term is negative  
 need  $m^6$  term to give stability

This describes a tricritical point where a line of  
 1<sup>st</sup> order transitions becomes a line of 2<sup>nd</sup> order  
 transitions

② put in spatially vary terms: ex: a superconductor  
 in an applied magnetic field. Order parameter is  
 condensate wavefunction  $\psi(\vec{r})$ .  
magnetic vector  
 $\downarrow$   
 potential

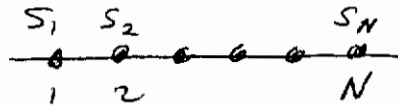
$$f(\psi, T) = f_0 + a|\psi|^2 + b|\psi|^4 + c |(\vec{\nabla} + i\vec{A})\psi|^2$$

minimize wrt  $\psi$  to get Abrikosov  
 vortex lattice

$\vec{\nabla} \times \vec{A} = \vec{B}$  magnetic field ↑  
kinetic energy of supercurrents

## Ising model in 1-dimension

$h=0$  for simplicity



$$\mathcal{H} = -J \sum_{i=1}^N s_i s_{i+1}$$

Define  $\sigma_i = s_i s_{i+1}$ ,  $i=1, \dots, N-1$

$$\sigma_i = \pm 1$$

$$\mathcal{H} = -J \sum_{i=1}^{N-1} \sigma_i \quad s_i s_j = \prod_{i=1}^{j-1} \sigma_i = (s_1 s_2)(s_2 s_3) \dots (s_{j-1} s_j) \\ = s_1 s_2^2 s_3^2 \dots s_{j-1}^2 s_j \\ = s_1 s_j$$

For every set of  $\{\sigma_i\}_{i=1}^{N-1}$  there are 2 possible spin configurations depending on whether  $s_1 = +1$  or  $-1$

For a given value of  $s_1$ , then

$$s_j = \frac{1}{s_1} \prod_{i=1}^{j-1} \sigma_i$$

So

$$Z = \sum_{\{s_i\}} e^{\beta J \sum_{i=1}^N s_i s_{i+1}} = 2 \sum_{\{\sigma_i\}} e^{\beta J \sum_{j=1}^{N-1} \sigma_j} = 2 \prod_{i=1}^{N-1} \sum_{\sigma_i = \pm 1} e^{\beta J \sigma_i}$$

↑  
two values for  $s_i$

$$Z = 2 \left[ \sum_{\sigma = \pm 1} e^{\beta J \sigma} \right]^{N-1} = 2 \left[ 2 \cosh \beta J \right]^{N-1}$$



## Gibbs free energy

$$G(h=0, T) = -k_B T \ln Z = -k_B T \ln 2 - k_B T (N-1) \ln (2 \cosh \beta J)$$

$$g = \lim_{N \rightarrow \infty} \frac{G}{N} = -k_B T \ln (2 \cosh \beta J)$$

entropy  $\Delta = - \left( \frac{\partial g}{\partial T} \right)_{h=0}$       specific heat  $C = T \left( \frac{\partial \Delta}{\partial T} \right)_{h=0}$   
at const  $h=0$        $= -T \left( \frac{\partial^2 g}{\partial T^2} \right)$

$$\Delta = k_B \ln (2 \cosh \beta J) + \frac{k_B T}{2 \cosh(\beta J)} \frac{\partial}{\partial T} [\cosh(\beta J)]$$

$$= k_B \ln (2 \cosh \beta J) + \frac{k_B T}{\cosh(\beta J)} \sinh(\beta J) J \frac{d\beta}{dT}$$

$$= k_B \ln (2 \cosh \beta J) - \frac{J}{T} \tanh \beta J$$

$$\Delta = k_B \left[ \ln (2 \cosh \beta J) - \beta J \tanh \beta J \right]$$

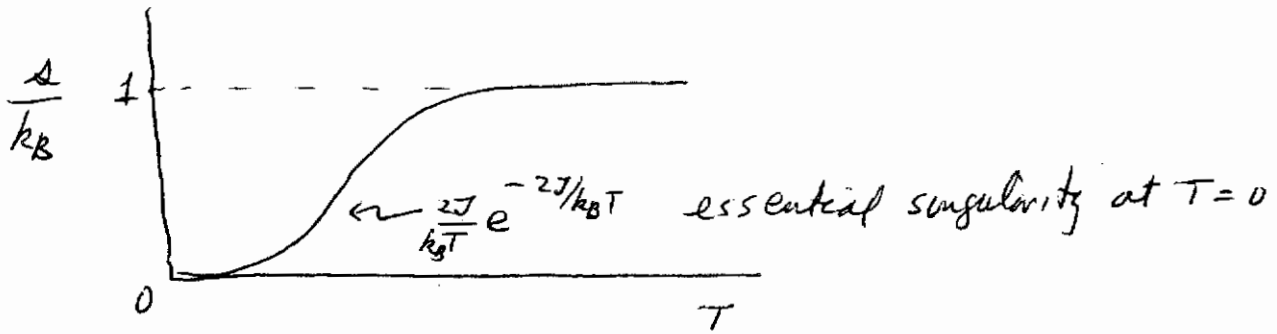
At  $T \rightarrow \infty$ ,  $\beta \rightarrow 0$ ,

$$\cosh \beta J \approx 1 + \frac{1}{2} (\beta J)^2$$
$$\tanh(\beta J) \approx \beta J$$
$$\Delta \approx k_B \left[ \ln [2 + (\beta J)^2] - (\beta J)^2 \right]$$
$$\approx k_B \ln 2$$

At  $T \rightarrow 0$ ,  $\beta \rightarrow \infty$

$$\cosh \beta J \approx e^{\beta J}$$
$$\tanh \approx \frac{1 - e^{-2\beta J}}{1 + e^{-2\beta J}} \approx 1 - 2e^{-2\beta J}$$

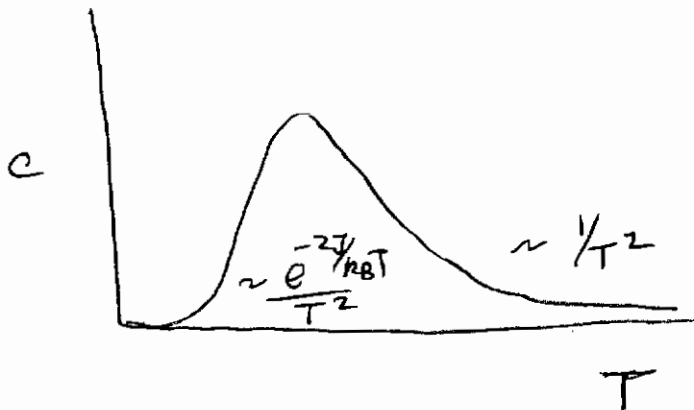
$$\Delta \approx k_B \left[ \ln e^{\beta J} - \beta J (1 - 2e^{-2\beta J}) \right] \approx \frac{2J}{T} e^{-2J/k_B T}$$



$$C = T \left( \frac{\partial a}{\partial T} \right) = k_B T \left\{ \frac{-2J \sinh \beta J}{2 \cosh \beta J} \frac{1}{k_B T^2} + \frac{J}{k_B T^2} \tanh \beta J + \frac{\beta J^2}{k_B T^2} \frac{\partial \tanh \beta J}{\partial (\beta J)} \right\}$$

$$= \frac{J^2}{k_B T^2} \frac{\partial (\tanh \beta J)}{\partial (\beta J)} = \frac{J^2}{k_B T^2} \frac{1}{(\cosh \beta J)^2}$$

$$C = k_B \left( \frac{\beta J}{\cosh \beta J} \right)^2$$



as  $T \rightarrow \infty$ ,  $\beta \rightarrow 0$

$$C \approx k_B \left( \frac{J}{k_B T} \right)^2$$

as  $T \rightarrow 0$ ,  $\beta \rightarrow \infty$

$$C \approx k_B \left( \frac{J}{k_B T} \right)^2 e^{-2J/k_B T}$$

essential singularity  
at  $T=0$

$\Rightarrow$  No singularity at any finite  $T$ .

$\Rightarrow$  No phase transition at any finite  $T$