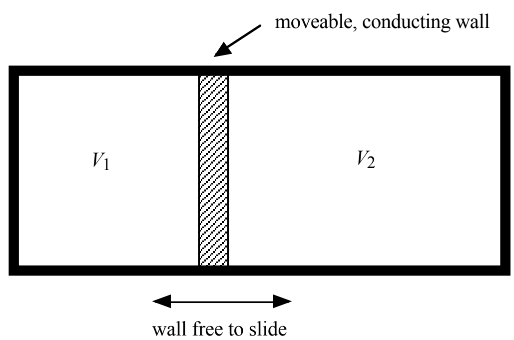
PHY 418 **Final Exam** **Spring 2012**

1) [20 points] In lecture we discussed the Debye model for the specific heat due to lattice vibrations in a crystalline solid. Prior to the Debye model, Einstein developed the following simpler model: each of the N atoms of the solid is viewed as sitting in a local isotropic harmonic potential with resonant frequency ω_0. In Einstein's model the atoms vibrate independently of each other. What is the specific heat at constant volume per particle, c_V, due to atomic vibrations in Einstein's model? Make a sketch of c_V vs temperature T. How does c_V behave in the high temperature limit, $k_B T \gg \hbar\omega_0$, and in the low temperature limit, $k_b T \ll \hbar\omega_0$?

2) [20 points] Consider a thermodynamic system consisting of two gases in volumes V_1 and V_2, separated by a thermally conducting, freely sliding wall, as shown in the diagram below. The gas in V_1 has N_1 particles, the gas in V_2 has N_2 particles, and particles cannot pass through the separating wall. The system is isolated from the rest of the universe, and the total volume $V_1 + V_2 = V$ is fixed. In thermal equilibrium, the pressures of the gases on the two sides of the sliding must be equal.



If the system is initially in thermal equilibrium at temperature T, for each case below, explain in which direction the wall between the two gases will move if the temperature is increased a small amount ΔT. If you know what you are doing, you can get the answer with a simple graphical analysis, no algebra needed! But however you do it, you must give a sound explanation for your answer, not just a guess!

a) The gas in V_1 is an ideal gas of fermions in the degenerate limit, i.e. $k_B T \ll \epsilon_F$. The gas in V_2 is a classical ideal gas. [7 pts]

b) The gas in V_1 is an ideal gas of bosons in a Bose-Einstein condensed state, i.e. $T < T_c$. The gas in V_2 is a classical ideal gas. [6 pts]

c) The gas in V_1 is an ideal gas of fermions, and the gas in V_2 is an ideal gas of bosons. However both are now in the non-degenerate limit where their behavior is approaching that of an ideal classical gas (i.e. keep the leading quantum correction to the classical equation of state). [7 pts]

3) [30 points] Consider a classical ideal gas of indistinguishable non-interacting molecules in equilibrium at temperature T, in the grand canonical ensemble with fugacity z. The particles may have arbitrary internal degrees of freedom (i.e. rotational, vibrational, or electronic). Express your answers to the questions below in terms of the fugacity z, the temperature T, and the single particle partition function Q_1.

a) Find an expression for the probability the system will have exactly N particles. [10 pts]

b) What is the average number of particles $\langle N \rangle$? [10 pts]

c) What is the fluctuation in the number of particles $\langle N^2 \rangle - \langle N \rangle^2$? [10 pts]

4) [30 points] Consider a degenerate *two-dimensional* gas of indistinguishable non-interacting spin 1/2 fermions of mass m in an external isotropic harmonic potential $V(\mathbf{r}) = \frac{1}{2} m \omega_0^2 |\mathbf{r}|^2$, where $\mathbf{r} = (x, y)$ (this might be a model for fermions in a magnetic trap). The quantized single particle energy levels are given by $\epsilon(n_x, n_y) = \hbar\omega_0(n_x + n_y + 1)$, where $n_x, n_y = 0, 1, 2, \ldots$ are integers. Assume the thermal energy is much greater than the spacing between the energy levels, i.e. $k_B T \gg \hbar\omega_0$.

a) If the number of single particle states between energies ϵ and $\epsilon + d\epsilon$ is $g(\epsilon)d\epsilon$, find the density of states $g(\epsilon)$. [6 pts]

b) What is the Fermi energy ϵ_F as a function of the number of particles N? [6 pts]

c) What is the total energy E of the gas, as a function of N, at $T = 0$? [6 pts]

d) Give an estimate for the spatial extent R of the Fermi gas about the origin of the harmonic potential. [6 pts]

e) Give a rough estimate of the specific heat of the gas C at low temperatures $T \ll T_F$. [6 pts]

Hint: Consider the shape of the contour of constant energy ϵ in the (n_x, n_y) plane.