Back to Ising model : Take $\tilde{f}(m,T,h) = f(m,T) - mh$

$$f(m,T) = f_0(T) + a(T)m^2 + b(T)m^4 + \cdots$$

↑ ignore higher order terms

Stability $\Rightarrow b(T) > 0$, $f(m,T)$ must have global minimum

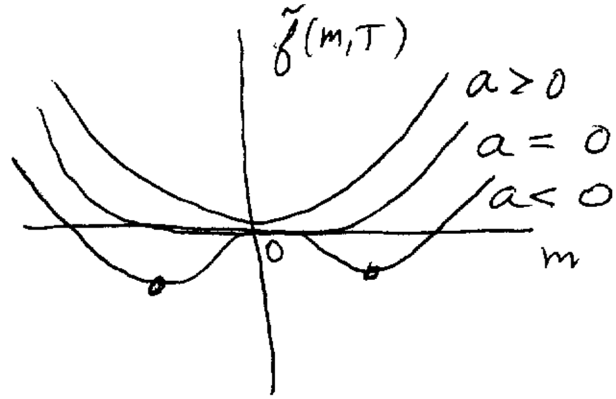
State of system is obtained by minimizing $\tilde{f}(m,T)$ with respect to m. Or equivalently, Gibbs free energy is

$$g(h,T) = \min_m \left[f(m,T) - hm \right]$$

On the 1st order line the ordering field $h=0$

$$\Rightarrow g(0,T) = \min_m \left[f(m,T) \right]$$

$\Rightarrow$ 2nd order critical point occurs when $a(T) = 0$



minimum of $\tilde{f}(m,T)$ increases **continuously** from $m=0$ as a decreases below zero.

When $a>0$ then $m=0$ minimizes $f(m,T)$
$\Rightarrow$ thermodynamic state has symmetry of H

When $a<0$, then $m = \pm m_0$ minimizes $f(m,T)$
$\Rightarrow$ thermodynamic state breaks symmetry

Expanding near T_c to lowest orders,

$$b(T) \approx b(T_c) \equiv b_0 \quad \text{a constant}$$

$$a(T) \simeq a_0[T-T_c] \quad a_0 \text{ a constant}$$

① <u>Behavior of order parameter near T_c</u>

$T < T_c$ minimize $f(m,T)$

$$\Rightarrow \quad 2am + 4bm^3 = 0$$

$$2a + 4bm^2 = 0 \quad \text{for } m \neq 0$$

$$m^2 = -\frac{a}{2b}$$

$$\boxed{m_0 = \pm\sqrt{\frac{a_0(T_c - T)}{2b_0}}} \propto |t|^\beta \quad \boxed{\beta = 1/2}$$

$$t = \left(\frac{T_c - T}{T_c}\right)$$

same β as found earlier

② <u>$h(m)$ curve at critical isotherm $T = T_c$</u>

$$g(h,T_c) = \min_m \left[f(m,T) - hm \right]$$

$$= \min_m \left[f_0 + b_0 m^4 - hm \right] \qquad a=0 \text{ at } T_c$$

$$\Rightarrow \quad 4b_0 m^3 - h = 0 \quad \Rightarrow \quad \boxed{h = 4b_0 m^3}$$

$$h \propto m^\delta \quad \boxed{\delta = 3} \qquad \text{same as before}$$

③ susceptibility $\chi = \frac{\partial m}{\partial h}$ at $h=0$

$$g(h,T) = \min_m \left[f(m,T) - hm \right]$$

$$\Rightarrow \quad 2am + 4bm^3 = h \qquad \text{"equation of state"}$$

$$\chi^{-1} = \frac{\partial h}{\partial m} = 2a + 12bm^2$$

$$\chi = \frac{1}{2a+12bm^2}$$

For $T > T_c$, $h=0 \Rightarrow m^2 = 0$

$$\boxed{\chi^+ = \frac{1}{2a}} = \frac{1}{2a_0(T-T_c)} \propto \frac{1}{|t|^{\gamma'}}, \quad \boxed{\gamma' = 1}$$

For $T < T_c$, $h=0 \Rightarrow m^2 = m_0^2 = \frac{-a}{2b} = \frac{a_0(T_c - T)}{2b_0}$

$$\chi^- = \frac{1}{2a_0(T-T_c) + \frac{12b_0 a_0}{2b_0}(T_c - T)}$$

$$\boxed{\chi^- = \frac{1}{4a_0(T_c - T)}} \propto \frac{1}{|t|^\gamma} \qquad \boxed{\gamma = 1}$$

$$\boxed{\lim_{T\to T_c} \frac{\chi^+}{\chi^-} = 2} \qquad \text{amplitude ratio}$$

all same as before

④ <u>specific heat at $h=0$</u> along 1st order transition line

from ① we have $m_0^2 = \frac{-a}{2b}$ $T<T_c$, $m_0^2 = 0$ $T>T_c$

$$\Rightarrow \quad g(h=0,T) = f(m_0,T) = f_0(T), \quad T>T_c$$

$$= f_0(T) + a\left(\frac{-a}{2b}\right) + b\left(\frac{-a}{2b}\right)^2, \quad T<T_c$$

$T<T_c$:
$$f(m_0,T) = f_0(T) - \frac{a^2}{2b} + \frac{a^2}{4b} = f_0(T) - \frac{a^2}{4b}$$

$$= f_0(T) - \frac{a_0^2}{4b_0}(T-T_c)^2$$

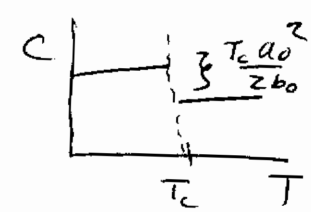
<u>specific heat</u>

$$s = -\frac{\partial g}{\partial T} \quad \Rightarrow \quad C = T\left(\frac{\partial s}{\partial T}\right)_{h=0} = -T\frac{\partial^2 g}{\partial T^2}$$

$$C = -T\frac{d^2 f(m_0(T),T)}{dT^2}$$

$$= \begin{cases} -T\dfrac{d^2 f_0}{dT^2} & T>T_c \\[2ex] -T\dfrac{d^2 f_0}{dT^2} + \dfrac{T a_0^2}{2b_0} & T<T_c \end{cases}$$

$$\Rightarrow \quad C(T\to T_c^-) - C(T\to T_c^+) = \frac{T_c a_0^2}{2b_0}$$



jump in specific heat at T_c

The piece $\frac{\partial^2 f_0}{\partial T^2}$ is the non singular piece of the specific heat. f_0 is the same as the "reference" free energy we used earlier when integrating the equation of state in the mean field or the van der Waals approx.

We can define a critical exponent α for the specific heat by $C \propto |t|^{-\alpha}$, or

$$\alpha = \lim_{t \to 0} \left[\frac{\ln C}{\ln |t|} \right]$$

For Landau theory this gives $\boxed{\alpha = 0}$

Summary: Landau theory = mean field theory

$h = 0$, $m_0(T) \sim |t|^\beta$ $\quad \beta = 1/2$

$T = T_c$, $h(m) \propto m^\delta$ $\quad \delta = 3$

$h = 0$, $\chi(T) \propto \frac{1}{|t|^\gamma}$ $\quad \gamma = 1$

$$\lim_{t \to 0} \frac{\chi^+}{\chi^-} = 2$$

$h = 0$, $C(T) \propto |t|^{-\alpha}$ $\quad \alpha = 0$

mean field critical exponents

exponent values in mean field approx are indep of dimension d.

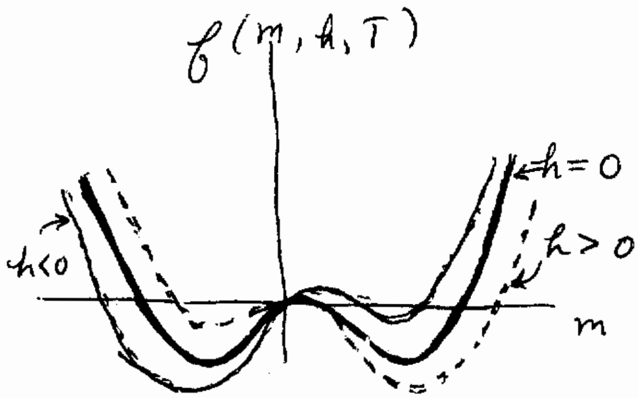
From exact solution of 2D Ising model

$\delta = 15$ $\quad \beta = 1/8$, $\gamma = 7/4$, $\underset{\text{log divergence}}{\alpha = 0}$, $\nu = 1$ $\quad C \propto \ln |t|$

Landau theory of 1st order transition

For $T < T_c$, $h \neq 0$

$$g(h,T) = \min_m \left[f(m,T) - hm \right] \equiv \min_m \left[f(m,h,T) \right]$$



as h goes smoothly through zero, the value of m that minimizes $f(m,h,T)$ jumps <u>discontinuously</u> from $+m_0$ to $-m_0$.

<u>2nd order</u> transition – order parameter goes <u>continuously</u> to zero

<u>1st order</u> transition – order parameter jumps <u>discontinuously</u>

Note: Landau theory ≡ mean field theory
gives the same values of the critical exponents independent of dimension d, and number of components of spin n.

For n-component spins with $\vec{m} = \frac{1}{N} \sum_i \vec{s}_i$

$$f(\vec{m},T) = f_0 + a|\vec{m}|^2 + b|\vec{m}|^4 + \cdots$$

everything comes out the same!

But can get some interesting new behaviors by doing other things.

① $f(m,T) = f_0 + a\, m^2 - b\, m^4 + c\, m^6$

$b > 0 \Rightarrow$ quartic term is negative
need m^6 term to give stability

This describes a tricritical point where a line of 1st order transitions becomes a line of 2nd order transitions

② put in spatially varying terms: ex: a superconductor in an applied magnetic field. Order parameter is condensate wave function $\psi(\vec{r})$.

magnetic vector potential

$$f(\psi,T) = f_0 + a|\psi|^2 + b|\psi|^4 + c\left|(\vec{\nabla} + i\vec{A})\psi\right|^2$$

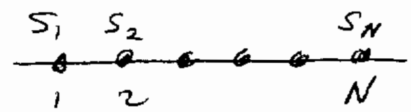
minimize wrt ψ to get Abrikosov vortex lattice

$\vec{\nabla}\times\vec{A} = \vec{B}$ magnetic field

kinetic energy of supercurrents

Ising model in 1-dimension

$h = 0$ for simplicity



$$\mathcal{H} = -J \sum_{i=1}^{N} s_i s_{i+1}$$

Define $\sigma_i = s_i s_{i+1}$, $i = 1, \cdots, N-1$

$$\sigma_i = \pm 1$$

$$\mathcal{H} = -J \sum_{i=1}^{N-1} \sigma_i \qquad s_1 s_j = \prod_{i=1}^{j-1} \sigma_i = (s_1 s_2)(s_2 s_3) \cdots (s_{j-1} s_j)$$

$$= s_1 s_2^2 s_3^2 \cdots s_{j-1}^2 s_j$$

$$= s_1 s_j$$

For every set of $\{\sigma_i\}_{i=1}^{N-1}$ there are 2 possible spin configurations depending on whether $s_1 = +1$ or -1
For a given value of s_1, then

$$s_j = \frac{1}{s_1} \prod_{i=1}^{j-1} \sigma_i$$

So

$$Z = \sum_{\{s_i\}} e^{\beta J \sum_{i=1}^{N} s_i s_{i+1}} = 2 \sum_{\{\sigma_i\}} e^{\beta J \sum_{j=1}^{N-1} \sigma_j} = 2 \prod_{j=1}^{N-1} \sum_{\sigma_i = \pm 1} e^{\beta J \sigma_i}$$

two values for s_1

$$Z = 2 \left[\sum_{\sigma = \pm 1} e^{\beta J \sigma} \right]^{N-1} = 2 \left[2 \cosh \beta J \right]^{N-1}$$

Gibbs free energy

$$G(h=0,T) = -k_B T \ln Z = -k_B T \ln 2 - k_B T (N-1) \ln(2\cosh \beta J)$$

$$g = \lim_{N\to\infty} \frac{G}{N} = -k_B T \ln(2\cosh\beta J)$$

entropy $s = -\left(\frac{\partial g}{\partial T}\right)_{h=0}$ specific heat at const $h=0$ $C = T\left(\frac{\partial s}{\partial T}\right)_{h=0}$

$$= -T\left(\frac{\partial^2 g}{\partial T^2}\right)$$

$$s = k_B \ln(2\cosh\beta J) + \frac{k_B T}{2\cosh(\beta J)} \frac{\partial}{\partial T}\left[\cosh(\beta J)\right]$$

$$= k_B \ln(2\cosh\beta J) + \frac{k_B T}{\cosh(\beta J)} \sinh(\beta J) J \frac{d\beta}{dT}$$

$$= k_B \ln(2\cosh\beta J) - \frac{J}{T}\tanh\beta J$$

$$s = k_B\left[\ln(2\cosh\beta J) - \beta J \tanh\beta J\right]$$

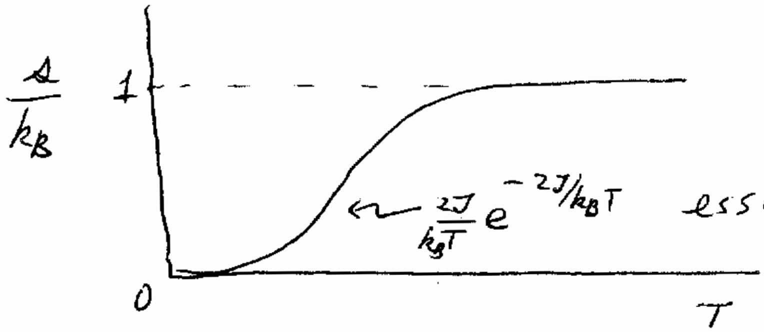
At $T\to\infty$, $\beta\to 0$,

$$\cosh\beta J \simeq 1 + \tfrac{1}{2}(\beta J)^2$$
$$\tanh(\beta J) \simeq \beta J$$
$$s \simeq k_B\left[\ln\left[2+(\beta J)^2\right] - (\beta J)^2\right]$$
$$\simeq k_B \ln 2$$

At $T\to 0$, $\beta\to\infty$

$$\cosh\beta J \simeq e^{\beta J}$$
$$\tanh \simeq \frac{1-e^{-2\beta J}}{1+e^{-2\beta J}} \simeq 1 - 2e^{-2\beta J}$$

$$s \simeq k_B\left[\ln e^{\beta J} - \beta J\left(1-2e^{-2\beta J}\right)\right] \simeq 2\frac{J}{T} e^{-2J/k_B T}$$



essential singularity at $T=0$

$$C = T\left(\frac{\partial s}{\partial T}\right) = k_B T \left\{ \cancel{\frac{-2J\sinh\beta J}{2\cosh\beta J}\,\frac{1}{k_B T^2}} + \cancel{\frac{J}{k_B T^2}\tanh\beta J} \right.$$

$$\left. + \frac{\beta J^2}{k_B T^2}\,\frac{\partial}{\partial(\beta J)}\tanh\beta J \right\}$$

$$= \frac{J^2}{k_B T^2}\,\frac{\partial}{\partial(\beta J)}\left(\tanh\beta J\right) = \frac{J^2}{k_B T^2}\,\frac{1}{(\cosh\beta J)^2}$$

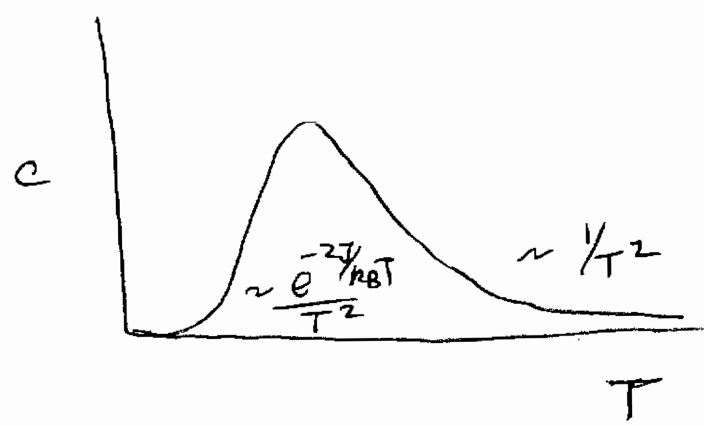
$$\boxed{C = k_B\left(\frac{\beta J}{\cosh\beta J}\right)^2}$$

as $T \to \infty$, $\beta \to 0$

$$C \simeq k_B\left(\frac{J}{k_B T}\right)^2$$

as $T \to 0$, $\beta \to \infty$

$$C \simeq k_B\left(\frac{J}{k_B T}\right)^2 e^{-2J/k_B T}$$



essential singularity at $T=0$

$\Rightarrow$ No singularity at any finite T.

$\Rightarrow$ No phase transition at any finite T