For free electrons where \( g(E) = C \sqrt{E} \)
\[ g'(E) = \frac{C}{2} \frac{1}{\sqrt{E}} \]

\[
\mu(T) = E_F - \frac{\pi^2}{6} (kB T)^2 \frac{1}{2 E_F} = E_F - \frac{\pi^2}{12} \frac{(kB T)^2}{E_F}
\]

\[
\mu(T) = E_F \left( 1 - \frac{1}{3} \left( \frac{\pi (kB T)^2}{2 E_F} \right) \right) = E_F \left( 1 - \frac{1}{3} \left( \frac{\pi T}{T_F} \right) \right)
\]

Correction is small for metals at room temp as \( T_F \approx 10000^\circ K \)

\( \Phi(E) = \int_0^\infty dE \, g(E) e^{m(E)} = \phi(E) = g(E) e^{m(E)} \)

\[
\mathcal{U} = \frac{E}{V} = \int_0^\mu \frac{dE}{E} \left( g(E) e + \frac{\pi^2 (k_B T)^2}{6} \left[ g(E) + \mu g'(E) \right] \right)
\]

\[
= \int_0^{E_F} \frac{dE}{E} \left( g(E) e + \frac{\pi^2 (k_B T)^2}{6} \left[ g(E) + \mu g'(E) \right] \right)
\]

Ground state energy density

\[
\mathcal{U}(T) = \mathcal{U}(0) + (\mu - E_F) g(E_F) e_F + \frac{\pi^2 (k_B T)^2}{6} \left[ g(E_F) + E_F g'(E_F) \right]
\]

\[
= \mathcal{U}(0) + \left[ -\frac{\pi^2}{6} \left( k_B T \right)^2 \frac{g'(E_F)}{g(E_F)} \right] g(E_F) e_F + \frac{\pi^2 (k_B T)^2}{6} \left[ g(E_F) + E_F g'(E_F) \right]
\]

\[
\mathcal{U}(T) = \mathcal{U}(0) + \frac{\pi^2 (k_B T)^2}{6} g(E_F)
\]
Specific heat per volume

\[ C_V = \frac{C_V}{V} = \frac{1}{V} \left( \frac{dE}{dT} \right)_V = \left( \frac{dU}{dT} \right)_V \]

\[ C_V = \frac{\pi^2}{3} \frac{k_B T}{g(E_F)} \]

For free electrons we can write \( g(E) = C \sqrt{E} \)

\[ m = \int_0^{E_F} g(E) \text{ d}E = \frac{2 \pi^2}{3} C \frac{E_F^{3/2}}{3} \Rightarrow C = \frac{3}{2} \frac{m}{E_F^{3/2}} \]

\[ g(E_F) = \frac{3}{2} \frac{m}{E_F^{3/2}} \cdot \text{density of states at Fermi energy} \]

\[ C_V = \frac{\pi^2}{2} \left( \frac{k_B T}{E_F} \right)^2 m k_B \]

or total specific heat \( C_V = V C_V \quad m V = N \)

\[ C_V = \frac{\pi^2}{2} \left( \frac{k_B T}{E_F} \right) N k_B \]

\[ \Rightarrow \text{specific heat due to Fermi gas of electrons in a conductor is } C_V \sim T \text{ at low temperatures} \]

We already saw that specific heat due to ionic vibrations (phonons) in a solid went like \( C_V \sim T^3 \)

at low temperatures (Debye model)

\[ \Rightarrow \text{electronic contribution to } C_V \text{ dominates at sufficiently low } T \]
Simple estimate of $C_V$

When increase temperature to $k_B T$, the electrons near the Fermi energy $E_F$ will increase their energy by an amount $\sim k_B T$. The number of such electrons—roughly per unit volume—is roughly

$$\frac{g(E_F)(k_B T)}{T} \approx \text{energy interval about } E_F \text{ of density of states}$$

density of states which increase get excited at $E_F$

$\Rightarrow$ increase in energy per unit volume is

$$\Delta U \sim \left( \frac{g(E_F)}{2} k_B T \right) (k_B T) \approx g(E_F)(k_B T)^2$$

$\Rightarrow C_V = \frac{\Delta U}{\Delta T} \approx g(E_F) k_B T = \frac{3}{2}\frac{M}{e^2} k_B T = \frac{3}{2}\frac{M k_B T}{14}$

The previous calculation gives the precise numerical coefficient
Electronic specific heat per volume

\[ C_V^{\text{elec}} = \frac{\pi^2}{2} \left( \frac{k_B T}{E_p} \right) N k_B \left( 1 + 0 \left( \frac{k_B T}{E_F} \right)^2 \right) \]

compared to classical result \( C_V^{\text{classical}} = \frac{N k_B}{V} \)

The correct result for degenerate Fermi gas is a factor

\[ \frac{\pi^2}{2} \left( \frac{k_B T}{E_F} \right) = \frac{\pi^2}{2} \left( \frac{T}{T_F} \right) \]

smaller than classical result by factor \( \sim 10^2 \) at room temperature.

Also, classical \( C_V \) is independent of \( T \), whereas Fermi-gas result is \( \propto T \).

At low \( T \), the conic contribution to \( C_V \) is

\[ C_V^{\text{con}} = \frac{12 \pi^4}{5} \left( \frac{T}{\Theta_D} \right)^3 N k_B \]

\[ \frac{C_{\text{elec}}}{C_{\text{con}}} = \frac{\pi^2}{2} \left( \frac{k_B T}{E_F} \right) \frac{5}{12 \pi^4} \left( \frac{\Theta_D}{T} \right)^3 \approx \frac{5}{24 \pi^2 \left( \frac{\Theta_D}{T_F} \right)} \left( \frac{\Theta_D}{T} \right)^2 \]

\[ \approx 1 \quad \text{when} \quad T^* = \sqrt{\frac{5}{24 \pi^2 \left( \frac{\Theta_D}{T_F} \right)}} \Theta_D \approx 0.15 \left( \frac{\Theta_D}{T_F} \right)^{1/2} \Theta_D \]

for metals, \( T_F \sim 10^4 \text{ }^\circ K, \quad \Theta_D \sim 10^2 \text{ }^\circ K \)

\[ T^* = 0.15 \sqrt{10^{-2}} \Theta_D \approx 0.015 \Theta_D \]

So conic contrib to \( C_V \) dominates over electronic contrib until \( T \ll 0.01 \Theta_D \) i.e. at \( 0(1) \text{ }^\circ K \). The electronic contrib dominates at lower temperatures.
Pauli paramagnetism of electron gas

\[ \vec{s} = \frac{1}{2} h \vec{\sigma} \]

An electron has intrinsic spin \( \vec{s} \) with intrinsic magnetic moment \( \vec{\mu} = -\mu_B \vec{s} \), where \( \mu_B = \frac{1}{2} \frac{e}{mc} \) is Bohr magneton.

In an external magnetic field \( \vec{B} \), there is an interaction energy \(-\vec{\mu} \cdot \vec{B} = \mu_B \sigma \vec{B} \), where \( \sigma = \pm 1 \) for spins parallel and antiparallel to \( \vec{B} \). The energy spectra for up or down electron spins becomes

\[ E_{\pm}(k) = E(k) \pm \mu_B B \]

where \( E(k) \) is spectrum at \( B = 0 \).

Since up and down electrons now have different energy spectra, we should treat them as two different populations of particles \( \Rightarrow \) they will be in equilibrium when their chemical potentials are equal, i.e., \( \mu_+ = \mu_- \).

This will induce a net magnetization in the system.

To see this, consider free electrons at \( T = 0 \).
When $\vec{B} = 0$, $E_+(\vec{k}) = E_-(\vec{k})$

Ground state occupations look as shown on the left. Equal numbers of $\uparrow$ and $\downarrow$ electrons

$m_+ = m_-$

When $\vec{B}$ is turned on, if there were no redistribution of electron spins, the situation would look like

Clearly the system can lower its energy by transferring $\uparrow$ electrons to $\downarrow$ electrons.

At equilibrium the system will look like

again the two populations have the same max energy $E_F$. But there are more $\downarrow$ electrons than $\uparrow$ electrons

Magnetization $\frac{M}{N} = -\mu_B (m_+ - m_-) > 0$

$\frac{M}{N}$ is parallel to $\vec{B} \Rightarrow$ paramagnetic effect
The calculation

Let \( g(\varepsilon) \) be the density of states when \( B = 0 \).

When \( B > 0 \), the density of states for \( \uparrow \) and \( \downarrow \) electrons are:

\[
\begin{align*}
g_\uparrow(\varepsilon + \mu_B B) &= \frac{1}{2} g(\varepsilon) \Rightarrow g_\uparrow(\varepsilon) = \frac{1}{2} g(\varepsilon - \mu_B B) \\
g_\downarrow(\varepsilon - \mu_B B) &= \frac{1}{2} g(\varepsilon) \quad g_\downarrow(\varepsilon) = \frac{1}{2} g(\varepsilon + \mu_B B)
\end{align*}
\]

The density of \( \uparrow \) and \( \downarrow \) electrons is then:

\[
m_\pm = \int_{-\infty}^{\infty} d\varepsilon \, g_\pm(\varepsilon) f(\varepsilon, \mu(B))
\]

where

\[
f(\varepsilon, \mu(B)) = \frac{1}{e^{(\varepsilon - \mu(B))/k_B T} + 1}
\]

\( \mu(B) \) is the chemical potential — it might depend on \( B \) — it is same for \( \uparrow \) and \( \downarrow \).

We will consider only the case that

\[
\mu_B B \ll \mu(B) \approx \varepsilon_F
\]

so spin interaction is small compared to \( \varepsilon_F \).
First we will show that

\[ \mu(B) = \mu(B=0) \left[ 1 + O\left( \frac{\mu B}{E_F} \right)^2 \right] \]

since we will work in the \( \mu B \ll E_F \) limit, we will then be able to ignore changes in \( \mu \) due to the finite \( B \), and just use \( \mu(B=0) \).

**Proof:**

Consider the total density of electrons

\[
m = m_+ + m_- = \int_{-\infty}^{\infty} d\varepsilon \ f(\varepsilon, \mu(B)) \left[ g_+(\varepsilon) + g_-(\varepsilon) \right]
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} d\varepsilon \ f(\varepsilon, \mu(B)) \left[ g_+(\varepsilon - \mu_B) + g_+(\varepsilon + \mu_B) \right]
\]

Shift integration variable \( \varepsilon - \mu_B \rightarrow \varepsilon \)

Shift integration variable \( \varepsilon + \mu_B \rightarrow \varepsilon \)

\[
= \frac{1}{2} \int_{-\infty}^{\infty} d\varepsilon \ g(\varepsilon) \left[ f(\varepsilon + \mu_B, \mu(B)) + f(\varepsilon - \mu_B, \mu(B)) \right]
\]

use fact that \( f(\varepsilon, \mu) \) depends only on \( \varepsilon - \mu \)

\[
m = \frac{1}{2} \int_{-\infty}^{\infty} d\varepsilon \ g(\varepsilon) \left[ f(\varepsilon, \mu(B) - \mu_B) + f(\varepsilon, \mu(B) + \mu_B) \right]
\]

expand \( f \) about \( \mu(B) \) for small \( \mu B \)
\[
m = \frac{1}{2} \int d\epsilon \, g(\epsilon) \left[ f(\epsilon, \mu(B)) - \frac{df}{d\mu} \mu B - \frac{1}{2} \frac{d^2f}{d\mu^2} \left( \mu B \right)^2 + \cdots \right]
\]
\[+ f(\epsilon, \mu(B)) + \frac{df}{d\mu} \mu_B B + \frac{1}{2} \frac{d^2f}{d\mu^2} \left( \mu_B B \right)^2 + \cdots \]

where derivatives above are evaluated at \( \mu = \mu(B) \).

The terms linear in \( B \) cancel!

\[
m = \int d\epsilon \, g(\epsilon) \left[ f(\epsilon, \mu(B)) + \frac{1}{2} \frac{d^2f}{d\mu^2} \left( \mu B \right)^2 + \cdots \right]
\]

If we ignored the \( (\mu B)^2 \) term the above would be

\[
m = \int d\epsilon \, g(\epsilon) \, f(\epsilon, \mu(B))
\]

But this is just the same formula we use to compute \( m \) at \( B = 0 \) ! The magnetic field \( B \) appears nowhere in the above, except via \( \mu(B) \).

Since the density is physically fixed by the sample and cannot change as one varies \( B \), we would conclude that

\[
\mu(B) = \mu(0) \text{ is independent of } B.
\]

Conclusion

This depends on our having ignored the \( (\mu B)^2 \) term, so we can expect

\[
\mu(B) = \mu(0) + \frac{(\mu_B B)^2}{\epsilon_F}
\]

where \( \epsilon_F \) appears on dimensional grounds.
To see this is so more explicitly, let's include the $(\mu B B)^2$ term and continue to calculate.

\[ m = \int \mathcal{d} \epsilon \, g(\epsilon) \left[ f(\epsilon, \mu(B)) + \frac{1}{2} \frac{d^2 f}{d\mu^2} (\mu B B)^2 \right] \]

Write $\mu(B) = \mu(B=0) + \delta \mu$ and expand in first term

\[ m = \int \mathcal{d} \epsilon \, g(\epsilon) \left[ f(\epsilon, \mu(B=0) + \delta \mu) + \frac{1}{2} \frac{d^2 f}{d\mu^2} (\mu B B)^2 \right] \]

\[ = \int \mathcal{d} \epsilon \, g(\epsilon) \, f(\epsilon, \mu(B=0)) \]

\[ + \int \mathcal{d} \epsilon \, g(\epsilon) \frac{df}{d\mu} \bigg|_{\mu = \mu(B=0)} \delta \mu \]

\[ + \frac{1}{2} \int \mathcal{d} \epsilon \, g(\epsilon) \frac{d^2 f}{d\mu^2} \bigg|_{\mu = \mu(B)} (\mu B B)^2 \]

The first term is just the density when $B=0$, i.e. $m$. Hence we get

\[ 0 = \int \mathcal{d} \epsilon \, g(\epsilon) \frac{df}{d\mu} \bigg|_{\mu = \mu(B=0)} \delta \mu + \frac{1}{2} \int \mathcal{d} \epsilon \, g(\epsilon) \frac{d^2 f}{d\mu^2} \bigg|_{\mu = \mu(B)} (\mu B B)^2 \]

So the correction to $\mu$ due to finite $B$ is

\[ \delta \mu = -\frac{1}{2} \frac{\int \mathcal{d} \epsilon \, g(\epsilon) \frac{d^2 f}{d\mu^2} \bigg|_{\mu = \mu(B)} (\mu B B)^2}{\int \mathcal{d} \epsilon \, g(\epsilon) \frac{df}{d\mu} \bigg|_{\mu = \mu(B=0)}} \]
To see how big the shift is, consider the limit $T \to 0$ where $\mu(B=0) = E_F$, and for a step function

\[
\frac{df}{d\mu} = -\frac{df}{de} = \delta(e - \mu)
\]

\[
\frac{d^2f}{d\mu^2} = \frac{d^2f}{de^2} = -\frac{d\delta(e - \mu)}{de}
\]

so

\[
\int d\epsilon g(\epsilon) \frac{df}{d\mu} \bigg|_{\mu = \mu(B=0)} = g(\mu(B=0)) = g(E_F)
\]

\[
\int d\epsilon g(\epsilon) \frac{d^2f}{d\mu^2} \bigg|_{\mu = \mu(B)} = g'(\mu(B)) (\mu_B B)^2
\]

\[
\delta \mu = -\frac{1}{2} \frac{g'(\mu(B)) (\mu_B B)^2}{g(E_F)}
\]

to lowest order, evaluate $g'(\mu(B))$ as $g'(E_F)$

The difference will only give higher order corrections of $O(\mu_B B)^4$

\[
\delta \mu = -\frac{g'(E_F)(\mu_B B)^2}{2g(E_F)}
\]

for free electrons with $g(\epsilon) = C\sqrt{\epsilon}$, so $g'(\epsilon) = \frac{1}{2} \frac{C}{\sqrt{\epsilon}}$ we get

\[
\delta \mu = -\frac{(\mu_B B)^2}{4E_F}
\]

so

\[
\mu(B) = E_F \left(1 - \frac{(\mu_B B)^2}{2E_F}\right)
\]
Now we compute

\[ \frac{M}{V} = -M_B (m^+ - m^-) = M_B (m^- - m^+) \]

\[ \frac{M}{V} = M_B \int_{-\infty}^{\infty} \text{d}e \ f(e, \mu) \left[ g - (e) - g + (e) \right] \]

\[ = M_B \int_{-\infty}^{\infty} \text{d}e \ f(e, \mu) \left[ \frac{1}{2} g (e + M_B B) - \frac{1}{2} g (e - M_B B) \right] \]

\[ = \frac{1}{2} M_B \int_{-\infty}^{\infty} \text{d}e \ g(e) \left[ f(e, \mu + M_B B) - f(e, \mu - M_B B) \right] \]

as before

\[ \exp \left[ f(e, \mu \pm M_B B) \right] = f(e, \mu) \pm \frac{df}{d\mu} M_B B \]

\[ \frac{M}{V} = \frac{1}{2} M_B \int_{-\infty}^{\infty} \text{d}e \ g(e) \left[ 2 \frac{df}{d\mu} M_B B \right] \]

\[ = M_B B \int_{-\infty}^{\infty} \text{d}e \ g(e) \left( -\frac{2f}{\text{d}e} \right) \quad \text{since} \quad \frac{df}{d\mu} = -\frac{2f}{\text{d}e} \]

To lowest order in temperature, \(-\frac{df}{\text{d}e} \approx \delta(e - \mu)\) with \(\mu = E_F\)

\[ \frac{M}{V} = M_B^2 B g(E_F) \]

Magnetic susceptibility

\[ \chi = \frac{\frac{2}{3} \chi_B}{2m} \]

Pauli susceptibility

\[ \chi_p = \frac{M_B^2 g(E_F)}{3} \]

For free-electron gas we earlier had

\[ g(E_F) = \frac{3}{2} \frac{m}{E_F} \]

\[ \Rightarrow \chi_p = M_B^2 \frac{3}{2} \frac{m}{E_F} \]

\(\chi_p > 0 \Rightarrow \text{paramagnetic}\)
Compare this to classical result. Average magnetization of a single spin is

\[
\langle m \rangle = \frac{\mu_b}{(\beta \mu_b B)^2} \left[ e^{-\beta \mu_b B} + e^{-\beta \mu_b B} \right]
\]

\[
\langle m \rangle = \mu_b \tanh (\beta \mu_b B)
\]

\[
\frac{M}{V} = \frac{\langle m \rangle N}{V} = \mu_b m \tanh (\beta \mu_b B)
\]

\[
\chi = \frac{d(M/V)}{dB}
\]

at low T \(\rightarrow 0\), \(\tanh (\beta \mu_b B) \rightarrow 1\), \(\frac{M}{V} \rightarrow \mu_b m\) all spins aligned.

Compare to quantum case:

\[
\frac{M}{V} = \frac{3}{2} \frac{m}{e_F} \mu_b^2 B
\]

smaller than classical result by factor \(\frac{3}{2} \frac{\mu_b^2 B}{e_F} \ll 1\)

at high T \(\beta \rightarrow 0\), \(\tanh (\beta \mu_b B) \rightarrow \beta \mu_b B\)

\[
\frac{M}{V} = \frac{\mu_b^2 B m}{k_B T}, \quad \chi = \frac{M_m}{k_B T} \sim \frac{1}{T}
\]

Compare to quantum case - at room temp, finite T corrections remain negligible all still

\[
\chi_p = \mu_b^2 \frac{3}{2} \frac{m}{e_F} \text{ independent of T}
\]

smaller than classical by factor \(\frac{3}{2} \left( \frac{k_B T}{e_F} \right) \ll 1\)