We said that to have a state of spontaneously broken symmetry at finite $T$, one needs to be in thermodynamic limit $N \to \infty$.

Similarly, true singular phase transitions can only occur in this $N \to \infty$ limit. Proof as follows:

**Partition function sum:**

$$Z(T, h) = \sum_{\{S_i\}} e^{-\beta H[\{S_i\}]}$$

For finite system ($N$ finite), the number of configurations to sum over is $2^N$ is finite.

$Z$ is therefore the sum of a finite number of analytic functions ("analytic" here in the sense of complex function theory — has no singularities as vary $T, h$). As such, $Z$ must itself be an analytic function of $T$ and $h$.

$\Rightarrow$ $Z$ can have no singularities

$\Rightarrow$ no singularities in any thermodynamic quantities

$\Rightarrow$ no phase transitions.

Only in thermodynamic limit of $N \to \infty$ is $Z$ now the sum of an infinite number of analytic functions. Such an infinite sum need NOT be analytic, so phase transitions can exist.
Having outlined what we might expect from the Ising model, we would now like to compute properties and see what happens.

However, an exact solution is not in general possible. Exact solutions to Ising model exist in:
- \(d = 1\) dimension - we will do this later
- \(d = 2\) dimension - famous solution by Onsager
- \(d = 3\) dimensions, the best one has in very accurate numerical simulations - no exact solution.

\[ H = -J \sum_{i,j} S_i S_j - h \sum_i S_i \]

\(\Rightarrow\) **Approximate Solution**

**Mean Field or Curie-Weiss Molecular Field Approximation**

Consider spin \(S_i\). Approximate the interaction of \(S_i\) with its neighbor \(S_j\) as an interaction with the thermal average value of \(\langle S_j \rangle \approx m\).

Instead of \(S_j\), seeing the specific \(S_j\) that vary from config to config, we say \(S_i\) sees only the effective average value of \(S_j\) - which is the same value \(m\) for all sites \(j\).

\[ H_{\text{MF}} = -J \sum_{i,j} S_i m - h \sum_i S_i \]

\(T\) sum over bonds
\[ H_{MF} = -\frac{\tilde{z}}{\tilde{a}} J \sum_i S_i m - \hbar \sum_i S_i \]
\[ = -\left(\frac{\tilde{z}}{2} J m + \hbar\right) \sum_i S_i \]

where \( \tilde{z} \) is the "coordination number" — the number of nearest neighbors of site \( i \). For a simple (3d) cubic lattice, \( \tilde{z} = 6 \). For a \( (2d) \) square lattice, \( \tilde{z} = 4 \).

In this approximation, the interaction of \( S_i \) with its neighbors is just like the interaction of \( S_i \) with an additional average magnetic field \( \frac{\tilde{z}}{2} J m \) — hence the origin of the name "mean field" approximation.

To complete the approx., we need to self-consistently compute \( m \) using \( H_{MF} \) and self-consistently solve for \( m \) from the resulting equation.

\[ H_{MF} = \sum_i H_{MF}^{(i)} \quad \text{where} \quad H_{MF}^{(i)} = -\left(\frac{\tilde{z}}{2} J m + \hbar\right) S \]

we have non-interacting spins in MF approx

\[ \Rightarrow \quad \langle S \rangle = m = \frac{\sum_s e^{-\beta H_{MF}^{(1)}} S}{\sum_s e^{-\beta H_{MF}^{(1)}}} \]

\[ = \frac{e^{-\beta(\frac{\tilde{z}}{2} J m + \hbar)} + e^{-\beta(\frac{\tilde{z}}{2} J m + \hbar)}}{e^{\beta(\frac{\tilde{z}}{2} J m + \hbar)} + e^{-\beta(\frac{\tilde{z}}{2} J m + \hbar)}} \]

\[ m = \tanh \left[ \beta \left(\frac{\tilde{z}}{2} J m + \hbar\right) \right] \quad \text{solves to get} \quad m(T, \hbar) \]

Note \( m(T, \hbar) = -m(T, -\hbar) \) as expected.
m = \tanh \left[ \frac{\beta z J m}{2} + \beta \theta \right]

Consider first \( \theta = 0 \)

\[ m = \tanh \left[ \frac{\beta z J m}{2} \right] \quad \text{could solve graphically} \]

Since \( \tanh x = x - \frac{1}{3} x^3 + O(x^5) \) we see that for \( \frac{\beta z J}{2} < 1 \), the only solution will be \( m = 0 \).

However for \( \frac{\beta z J}{2} > 1 \) there are two additional solutions \( m = \pm m_0 \)

\[ \tanh \left( \frac{\beta z J m}{2} \right) \]

\[ \frac{\beta z J}{2} > 1 \]

\[ \frac{\beta z J}{2} < 1 \]

\[ \Rightarrow \text{Critical temperature} \quad k_B T_c = \frac{z J}{2} \]

\[ T > T_c \Rightarrow m = 0 \]

\[ T < T_c \Rightarrow m = \pm m_0 \quad m = 0 \ u \ \text{unstable solution} \]
For $T < T_c$, $m=0$ is unstable. $m = \pm m_0$ are the stable solutions. To see this:

$$m = \tanh \left( \frac{\beta J m + \beta h}{2} \right)$$

$$h = \frac{1}{\beta} \tanh^{-1} m - \frac{\pi J m}{2}$$

$$\left( \frac{df}{dm} \right)_T = k \Rightarrow f(m, T) = \int_0^m a(m') dm' + f(0, T)$$

Therefore we can plot the free energy $f(m, T) - f(0, T)$.

- Gibbs free energy
  $$g(n=0, T) = \min_m f(m, T) \Rightarrow m = \pm m_0$$

$m_0$ gives the min of the free energy and so is the equilibrium solution.
For $T > T_c$, the situation looks like

$$\begin{array}{l}
\text{now } \int h(m') \, dm' \text{ is the positive of the area shown to the left - it increases monotonically as } m \text{ increases.}
\end{array}$$

so the free energy looks like

$$\begin{array}{l}
\Rightarrow m = 0 \text{ is min of } f(m, T)
\end{array}$$

Recall - the plots of $f(m, T)$ shown above for $T > T_c$ and $T < T_c$ are exactly the same as we saw in discussing the van der Waals theory of the liquid-gas phase transition!
We can examine these points analytically if we consider behavior near \( T_c \) where \( m \) is small. This analysis will introduce the \textit{critical} exponents \( \beta, \gamma, \delta \) that characterize the critical point at \( (T_c, \mu = 0) \).

\[
\frac{\mu}{2} = k_B T_c \quad \text{and} \quad \tanh x \approx x - \frac{1}{3} x^3 \quad \text{for small} \quad x.
\]

For small \( \frac{\mu}{k_B T} \), near \( T_c \) where \( m \) is small, expand the taut:

\[
m = \left( \frac{T_c}{T} m + \frac{\mu}{k_B T} \right) - \frac{1}{3} \left( \frac{T_c}{T} m + \frac{\mu}{k_B T} \right)^3
\]

\[
\text{for small} \quad \frac{\mu}{k_B T} \ll m.
\]

\[
m = \left( \frac{T_c}{T} m + \frac{\mu}{k_B T} \right) - \frac{1}{3} \left( \frac{T_c}{T} \right)^3 m^3 - \frac{\left( \frac{T_c}{T} \right)^2 m^2}{k_B T} \frac{\mu}{k_B T}
\]

\[
m(1 - \frac{T_c}{T}) + \frac{1}{3} \left( \frac{T_c}{T} \right)^3 m^3 = \frac{\mu}{k_B T} \left( 1 - \frac{T_c}{T} \right)^2 m^2
\]

\[
h = k_B T \left\{ \frac{m(1 - \frac{T_c}{T}) + \frac{1}{3} \left( \frac{T_c}{T} \right)^3 m^3}{1 - \left( \frac{T_c}{T} \right)^2 m^2} \right\}
\]

\[
(\star) \quad \sqrt{h} \approx \frac{k_B T}{3} \left\{ m(1 - \frac{T_c}{T}) + \left( 1 - \frac{T_c}{T} \right) \left( \frac{T_c}{T} \right)^2 + \frac{1}{3} \left( \frac{T_c}{T} \right)^3 m^3 \right\}
\]

\[
(\star \star) \quad \frac{\mu}{k_B T_c} \approx m^3 \quad \text{or} \quad m \propto h^{1/3}
\]

At \( T = T_c \) \textit{critical isotherm}:

\[
h = \frac{k_B T_c}{3} m^3 \quad \propto m^\delta \quad \delta = 3
\]

\[
\text{or} \quad m \propto h^{1/3}
\]
(2) At \( h = 0 \) on coexistence line from (**) with \( h = 0 \) we have

\[
(1 - \frac{T_c}{T}) m + \left[ \frac{1}{3} \left( \frac{T_c}{T} \right)^3 + \frac{1}{3} \right] m^3 = 0
\]

as \( T \to T_c^- \), \( (1 - \frac{T_c}{T}) + \frac{1}{3} m^2 = 0 \)

\[
m = \pm \sqrt{\frac{3(T_c - T)}{T}}
\]

Define \( t = \frac{T_c - T}{T_c} \)

\[m \propto \sqrt{3t} \propto t^{\frac{3}{2}} \quad \beta = \frac{3}{2}\]

(3) At \( h = 0 \) on coexistence line as \( T \to T_c \)

from (**)\)

\[
\frac{\partial h}{\partial m} = k_B T \left\{ (1 - \frac{T_c}{T}) + 3 \left[ (1 - \frac{T_c}{T}) \frac{T_c}{T} \right]^2 + \frac{1}{3} \left( \frac{T_c}{T} \right)^3 \right\} m^2
\]

\[
= k_B T \left\{ (1 - \frac{T_c}{T}) + m^2 \right\} \quad \text{as} \ T \to T_c
\]

As \( T \to T_c^+ \) from above, \( m = 0 \)

\[
\frac{\partial h}{\partial m} = k_B T \left( 1 - \frac{T_c}{T} \right) = k_B (T - T_c)
\]

\[
\text{magnetic susceptibility} \Rightarrow \frac{\partial m}{\partial \chi} = \chi^* = \frac{1}{k_B (T - T_c)} \propto \frac{1}{|T|^{\gamma}}, \quad \gamma = 1
\]

Note: at high temp \( T \to T_c \), \( \chi \propto \frac{1}{T} \) just like in Curie paramagnetism. Hence we say the \( T > T_c \) phase is paramagnetic.