PHY 418 **Final Exam** **Spring 2018**

Answer all questions. Please put a box around your final answer for each part. Cross out anything you don't want me to look at. On the inside cover of your blue book, please write the academic honesty pledge: "I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own."

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1) [25 points]

Consider a system of particles in the grand canonical ensemble, with μ the chemical potential and z the fugacity. If $\langle N\rangle$ is the average number of particles and $\langle E\rangle$ is the average total energy,

a) [15 pts] Show that

$$\frac{1}{T}\left(\frac{\partial\langle E\rangle}{\partial\mu}\right)_{T,V} = \frac{1}{k_B T^2}\left[\langle EN\rangle - \langle E\rangle\langle N\rangle\right]$$

b) [10 pts] Show that

$$\left(\frac{\partial\langle N\rangle}{\partial T}\right)_{z,V} = \frac{1}{T}\left(\frac{\partial\langle E\rangle}{\partial\mu}\right)_{T,V}$$

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2) [25 points]

Consider photons of a given energy $\varepsilon = \hbar\omega$.

a) [11 pts] If $\langle n\rangle$ is the average number of such photons in equilibrium at temperature T, show that the fluctuation in the number of photons is

$$\langle n^2\rangle - \langle n\rangle^2 = -\frac{1}{\varepsilon}\frac{d\langle n\rangle}{d\beta}, \qquad \text{where } \beta = 1/k_B T$$

b) [10 pts] Using the formula for the equilibrium value of $\langle n\rangle$, apply the above result to determine the relative fluctuation in the number of photons,

$$\frac{\langle n^2\rangle - \langle n\rangle^2}{\langle n\rangle^2}.$$

Is this large or small?

c) [4 pts] Can photons undergo Bose-Einstein condensation? You must give an explanation for your answer.

3) [25 points]

Consider a gas of N indistinguishable, non-relativistic, non-interacting spin 1/2 fermions (spin degeneracy $g_s = 2$), confined to move in a one dimensional system of length L.

a) [10 pts] The density of states $g(\varepsilon)$ is defined as the number of single-particle energy levels with energy ε, per unit energy per unit length. Compute $g(\varepsilon)$ for this fermi gas.

b) [6 pts] Compute the Fermi energy of the gas as a function of its density $n = N/L$.

c) [9 pts] Compute the pressure of the gas at $T = 0$ as a function of its density n.

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4) [25 points]

Consider an ideal Bose gas of free, non-relativistic particles in a box in 3 dimensions. The particles have an internal degree of freedom that can take only one of two energy values, the ground state with $\varepsilon_0 = 0$, and an excited state with $\varepsilon_1 > 0$. Determine the Bose-Einstein condensation temperature T_c of the gas as a function of ε_1. You only need to write the equation by which one could in principle solve for T_c.

Then, for the particular case that $\varepsilon_1/k_B T \gg 1$, show that,

$$\frac{T_c}{T_{c0}} = 1 - \frac{2\mathrm{e}^{-\varepsilon_1/k_B T_{c0}}}{3\,\zeta(\frac{3}{2})}$$

where T_{c0} is the transition temperature when ε_1 is infinite, and ζ is the Riemann zeta function.

Hint: Recall that the standard functions for bosons are

$$g_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty dy\, \frac{y^{n-1}}{z^{-1}e^y - 1} = \sum_{\ell=1}^\infty \frac{z^\ell}{\ell^n} \qquad \text{and that} \qquad \Gamma(n+1) = n\Gamma(n) \text{ and } \Gamma(1/2) = \sqrt{\pi}$$