Black Body Radiation

Cavity radiation — a volume V at fixed temp T absorbs + emits electromagnetic radiation. What are characteristics of this equilib radiation at fixed T?

EM waves with wave vector $\vec{k}$, freq $\omega = c|\vec{k}|$
two transverse polarizations for each $\vec{k}$.

Regard each mode as an oscillator. If excited to energy level n, the energy in the oscillator is

$$\epsilon = n\hbar\omega = n\hbar c k \Rightarrow n \text{ "photons" in this mode}$$

average energy in a given mode is therefore

$$\langle \epsilon \rangle = \hbar\omega \langle n \rangle = \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}$$

(ignore ground state energy $\frac{1}{2}\hbar\omega$ as it is T-indep constant)

For a volume $V = L^3$, periodic boundary conditions give the allowed wave vectors $\vec{k} = \frac{2\pi}{L}\vec{m}$ $\quad m_x, m_y, m_z$ integers

Density of states $g(\omega)$

two polarizations for each $\vec{k}$

$$\int g(\omega)\,d\omega = 2\sum_{\vec{k}} = \frac{2V}{(2\pi)^3}\int d^3k$$

$$\Rightarrow g(\omega)\,d\omega = \frac{2V}{(2\pi)^3} 4\pi k^2 dk = \frac{V}{\pi^2}\frac{\omega^2 d\omega}{c^3}$$

using $k = \omega/c$

$$g(\omega) = \frac{V\omega^2}{\pi^2 c^3}$$

average energy per volume at freq ω is

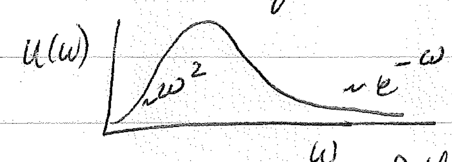
\# modes at freq ω

average energy in a given mode at freq ω

$$u(\omega) = \frac{g(\omega)}{V}\left(\frac{\hbar\omega}{e^{\beta\hbar\omega}-1}\right)$$

$$u(\omega) = \frac{\hbar\,\omega^3}{\pi^2 c^3\,(e^{\beta\hbar\omega}-1)}$$

← Black Body Spectrum
Planck's formula



Pathria fig 7.7

<u>Total energy density</u>

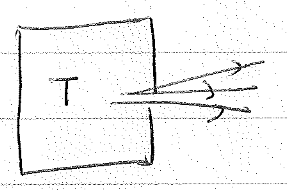
$$\frac{U}{V} = \int_0^\infty u(\omega)\,d\omega = \frac{\hbar}{\pi^2 c^3}\int_0^\infty d\omega\,\frac{\omega^3}{e^{\beta\hbar\omega}-1}$$

$$= \frac{\hbar}{\pi^2 c^3}\frac{1}{(\beta\hbar)^4}\underbrace{\int_0^\infty dx\,\frac{x^3}{e^x-1}}_{\frac{\pi^4}{15}} \qquad x = \beta\hbar\omega$$

$$\frac{U}{V} = \left(\frac{\pi^2 k_B^4}{15\,\hbar^3 c^3}\right)T^4$$

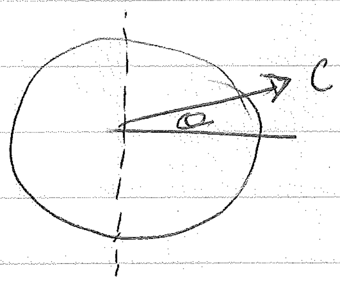
<u>Note</u>: A big difference between photons and phonons is that for phonons there is a largest possible $|\vec{k}|$ set by the spacing between the atoms in the lattice. For photons there is no such maximum $|\vec{k}|$.

energy flux from a cavity, exiting from a hole



$$\text{flux} \quad \mathcal{F} = \left(\frac{U}{V}\right) c \quad \langle \cos\theta \rangle$$

energy density — speed — projection of velocity in outwards direction



$$\langle \cos\theta \rangle = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \, \sin\theta \cos\theta$$

← only in outward direction

$$= \frac{2\pi}{4\pi} \left(\frac{\sin^2\theta}{2} \right)_0^{\pi/2} = \frac{1}{4}$$

$$\mathcal{F} = \left(\frac{U}{V}\right) \frac{c}{4} = \sigma T^4$$

← Stefan Boltzmann Law

$$\text{where} \quad \sigma = \frac{\pi^2 k_B^4}{60 \, \hbar^3 c^2} = 5.7 \times 10^{-8} \, \frac{W}{m^2 \cdot {}^\circ K^4}$$

Stefan's constant

We also have

$$\frac{pV}{k_B T} = \ln \mathcal{Z} = -\sum_{k} 2 \ln\left(1 - e^{-\beta \epsilon_k}\right)$$

polarizations

for BE with $\mu = 0$

$$= -\frac{2V}{(2\pi)^3} \int dk \, 4\pi k^2 \ln\left(1 - e^{-\beta \hbar c k}\right)$$

$$= -\int_0^\infty d\omega \, g(\omega) \ln\left(1 - e^{-\beta \hbar \omega}\right)$$

$$= -\frac{V}{\pi^2 c^3} \int_0^\infty d\omega \, \omega^2 \ln\left(1 - e^{-\beta \hbar \omega}\right)$$

integrate by parts

$$\frac{pV}{k_B T} = \frac{-V}{\pi^2 c^3}\underbrace{\left[\frac{\omega^3}{3}\ln(1-e^{-\beta\hbar\omega})\right]_0^\infty}_{=\,0} + \frac{V}{\pi^2 c^3}\int_0^\infty d\omega\,\frac{\omega^3}{3}\,\frac{\beta\hbar e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}}$$

$$\frac{pV}{k_B T} = \frac{V\beta\hbar}{3\pi^2 c^3}\int_0^\infty d\omega\left(\frac{\omega^3}{e^{\beta\hbar\omega}-1}\right)$$

compare with computation of $\frac{U}{V}$

$$= \frac{\beta}{3}U = \frac{1}{3}\frac{U}{k_B T}$$

$$\Rightarrow \boxed{\frac{1}{3}U = PV} \qquad \text{pressure of photon gas}$$

compare to non relativistic ideal gas

$U = \frac{3}{2}Nk_BT$, $PV = Nk_BT \Rightarrow \frac{2}{3}U = PV$

The previous examples of phonons in a solid and Black Body radiation were problems involving <u>bosons</u> with excitation spectrum

$\epsilon_k = \hbar\omega_k = \hbar c|\vec{k}|$ (ie linear spectrum)

and zero chemical potential $\mu = 0$.

non-interacting

Now we want to turn to the problem of an ideal quantum gas (bosons <u>or</u> fermions) of physical particles with ~~an excitation~~ an ordinary non-relativistic excitation spectrum

$$\epsilon_k = \frac{\hbar^2 k^2}{2m} \quad \text{(ie quadratic spectrum)}$$

and $\mu \neq 0$.

Ideal Quantum Gas — Grand canonical ensemble

$$\ln \mathcal{Z} = \pm \sum_i \ln\left(1 \pm e^{-\beta(\epsilon_i - \mu)}\right) \qquad + \text{FD}, \; - \text{BE}$$

for free particles, states can be labeled by ~~wavevector~~ wavevector $\vec{k}$ with $k_\mu = \frac{2\pi n_\mu}{L}$, $n_\mu = 0, \pm 1, \pm 2, \ldots$ due to periodic boundary conditions. volume $V = L^3$

$$\Rightarrow \underbrace{\sum_i}_{\text{states}} \rightarrow \sum_s \sum_{\vec{k}} \rightarrow g_s \frac{V}{(2\pi)^3} \int_0^\infty dk \, 4\pi k^2$$

spin polarizations # spin states for each $\vec{k}$

for free particles, ϵ depends only on $|\vec{k}|$. Define density of states $g(\epsilon)$ such that

$$\frac{g_s}{(2\pi)^3} \int_{k_1}^{k_2} dk \, 4\pi k^2 = \int_{\epsilon_{k_1}}^{\epsilon_{k_2}} g(\epsilon) \, d\epsilon$$

$g(\epsilon)$ = # states with energy ϵ per unit energy per volume

$$\Rightarrow g(\epsilon) = \frac{g_s 4\pi}{(2\pi)^3} k^2 \frac{dk}{d\epsilon}$$

For non-relativistic particles $\epsilon = \frac{\hbar^2 k^2}{2m}$, $k = \sqrt{\frac{2m\epsilon}{\hbar^2}}$

$$g(\epsilon) = \frac{g_s 4\pi}{(2\pi)^3} \frac{2m\epsilon}{\hbar^2} \sqrt{\frac{2m}{\hbar^2}} \frac{1}{2\sqrt{\epsilon}}$$

$$= \frac{2\pi g_s}{(2\pi)^3} \left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{\epsilon} = \left(\frac{2\pi m}{h^2}\right)^{3/2} \frac{2^{3/2} (2\pi)^{3/2}}{(2\pi)^2} g_s \sqrt{\epsilon}$$

Density of States

$$\boxed{g(\epsilon) = \left(\frac{2\pi m}{h^2}\right)^{3/2} \frac{2 g_s}{\sqrt{\pi}} \sqrt{\epsilon}}$$

$g \sim \sqrt{\epsilon}$

or $\quad g(\epsilon) = \frac{2g_s}{\sqrt{\pi}\,\lambda^3}\frac{1}{k_BT}\sqrt{\frac{\epsilon}{k_BT}}$ $\quad$ using $\lambda = \left(\frac{h^2}{2\pi m k_B T}\right)^{1/2}$

pressure

$$\frac{p}{k_BT} = \frac{1}{V}\ln \mathcal{L} = \pm\frac{1}{V}\sum_i \ln\left(1 \pm z\,e^{-\beta\epsilon_i}\right) \qquad z = e^{\beta\mu}$$

$$= \pm\int_0^\infty d\epsilon\; g(\epsilon)\ln\left(1 \pm z e^{-\beta\epsilon}\right)$$

$$= \pm\left(\frac{2\pi m}{h^2}\right)^{3/2}\frac{2g_s}{\sqrt{\pi}}\int_0^\infty d\epsilon\,\sqrt{\epsilon}\,\ln\left(1 \pm z e^{-\beta\epsilon}\right)$$

substitute variables $y = \beta\epsilon$

$$\frac{p}{k_BT} = \pm\left(\frac{2\pi m k_B T}{h^2}\right)^{3/2}\frac{2g_s}{\sqrt{\pi}}\int_0 dy\; y^{1/2}\ln\left(1 \pm z e^{-y}\right)$$

integrate by parts $\qquad \lambda = \left(\frac{h^2}{2\pi m k_B T}\right)^{1/2}$ thermal wavelength

$$\frac{p}{k_BT} = \pm\frac{2g_s}{\sqrt{\pi}\,\lambda^3}\left\{\underbrace{\frac{2}{3}y^{3/2}\ln\left(1 \pm z e^{-y}\right)\Big|_0^\infty}_{=0} - \int_0^\infty dy\,\frac{2}{3}y^{3/2}\frac{(\mp z e^{-y})}{1 \pm z e^{-y}}\right\}$$

$$\boxed{\frac{p}{k_BT} = \frac{4g_s}{3\sqrt{\pi}\,\lambda^3}\int_0^\infty dy\,\frac{y^{3/2}}{z^{-1}e^{y} \pm 1}}$$

+ FD
− BE

density of particles $\quad \frac{N}{V} = \frac{1}{V}\sum_i \langle n_i \rangle$

$$\frac{N}{V} = \frac{1}{V}\sum_i \frac{1}{z^{-1}e^{\beta\epsilon_i} \pm 1} = \int_0^\infty d\epsilon\; g(\epsilon)\frac{1}{z^{-1}e^{\beta\epsilon} \pm 1}$$

$$= \left(\frac{2\pi m}{h^2}\right)^{3/2}\frac{2g_s}{\sqrt{\pi}}\int_0^\infty d\epsilon\,\frac{\sqrt{\epsilon}}{z^{-1}e^{\beta\epsilon} \pm 1}$$

$$= \left(\frac{2\pi m k_B T}{h^2}\right)^{3/2}\frac{2g_s}{\sqrt{\pi}}\int dy\,\frac{y^{1/2}}{z^{-1}e^{y} \pm 1}$$

$$\boxed{\frac{N}{V} = \frac{2g_s}{\sqrt{\pi}\,\lambda^3}\int_0^\infty dy\,\frac{y^{1/2}}{z^{-1}e^{y} \pm 1}}$$

+ FD
− BE

energy density $\quad E = \sum_i \epsilon_i \langle n_i \rangle$

$$\frac{E}{V} = \frac{1}{V} \sum_i \frac{\epsilon_i}{z^{-1} e^{\beta \epsilon_i} \pm 1} = \int_0^\infty d\epsilon \, g(\epsilon) \frac{\epsilon}{z^{-1} e^{\beta\epsilon} \pm 1}$$

$$= \frac{2 g_s}{\sqrt{\pi}\,\lambda^3} k_B T \int_0^\infty dy \frac{y^{3/2}}{z^{-1} e^{y} \pm 1}$$

$$\frac{E}{V} = \frac{3}{2} k_B T \, \frac{4 g_s}{3\sqrt{\pi}\,\lambda^3} \int_0^\infty \frac{y^{3/2}}{z^{-1} e^{y} \pm 1} = \left(\frac{3}{2} k_B T\right)\left(\frac{p}{k_B T}\right)$$

$$\Rightarrow \frac{E}{V} = \frac{3}{2} p \quad \text{or} \quad \boxed{p = \frac{2}{3} \frac{E}{V}}$$

both fermions and bosons (same result as for classical ideal gas!!) nonrelativistic only

Define "standard functions" (see Pathria Appendices D and E)

$$f_n(z) \equiv \frac{1}{\Gamma(n)} \int_0^\infty dy \frac{y^{n-1}}{z^{-1} e^{y} + 1} = \sum_{\ell=1}^{\infty} (-1)^{\ell+1} \frac{z^\ell}{\ell^n}$$

$$g_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty dy \frac{y^{n-1}}{z^{-1} e^{y} - 1} = \sum_{\ell=1}^{\infty} \frac{z^\ell}{\ell^n}$$

$\Gamma(n+1) = n\,\Gamma(n)$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$\Rightarrow \Gamma(3/2) = \frac{1}{2}\sqrt{\pi}$

$\Gamma(5/2) = \frac{3}{4}\sqrt{\pi}$

In terms of these:

Fermions	Bosons
$\frac{p}{k_B T} = \frac{g_s}{\lambda^3} f_{5/2}(z)$	$\frac{p}{k_B T} = \frac{g_s}{\lambda^3} g_{5/2}(z)$
$\frac{N}{V} = \frac{g_s}{\lambda^3} f_{3/2}(z)$	$\frac{N}{V} = \frac{g_s}{\lambda^3} g_{3/2}(z)$
$\frac{E}{V} = \frac{3}{2} k_B T \frac{g_s}{\lambda^3} f_{5/2}(z)$	$\frac{E}{V} = \frac{3}{2} k_B T \frac{g_s}{\lambda^3} g_{5/2}(z)$
$\frac{E}{N} = \frac{3}{2} k_B T \frac{f_{5/2}(z)}{f_{3/2}(z)}$	$\frac{E}{N} = \frac{3}{2} k_B T \frac{g_{5/2}(z)}{g_{3/2}(z)}$