Equation of state : low densities ~ virial expansion

$z \ll 1$ "non-degenerate"

← keep lowest terms in series expansion

$$\frac{p}{k_B T} = \frac{g_s}{\lambda^3} \left\{ \begin{array}{c} f_{5/2} \\ g_{5/2} \end{array} \right\} = \frac{g_s}{\lambda^3} \left(z \mp \frac{z^2}{2^{5/2}} + \cdots \right) \quad \begin{array}{l} - \text{FD} \\ + \text{BE} \end{array}$$

$$\frac{N}{V} = \frac{g_s}{\lambda^3} \left\{ \begin{array}{c} f_{3/2} \\ g_{3/2} \end{array} \right\} = \frac{g_s}{\lambda^3} \left(z \mp \frac{z^2}{2^{3/2}} + \cdots \right)$$

$$\Rightarrow \frac{p}{k_B T} = \frac{N}{V} \frac{\left(z \mp \frac{z^2}{2^{5/2}} + \cdots \right)}{\left(z \mp \frac{z^2}{2^{3/2}} + \cdots \right)} = \frac{N}{V} \left(1 \mp \frac{z}{2^{5/2}} + \cdots \right) \left(1 \pm \frac{z}{2^{3/2}} + \cdots \right)$$

$$= \frac{N}{V} \left(1 \pm \frac{z}{2^{3/2}} \mp \frac{z}{2^{9/2}} + \cdots \right) \qquad \frac{1}{2^{3/2}} - \frac{1}{2^{5/2}} = \frac{2}{2^{5/2}} - \frac{1}{2^{5/2}} = \frac{1}{2^{5/2}}$$

$$pV = N k_B T \left(1 \pm \frac{z}{2^{5/2}} + \cdots \right)$$

↑ quantum correction to classical ideal gas law.

+ FD — p increases compared to classically
   — effective repulsion due to Pauli exclusion

— BE — p decreases compared to classically
   — effective attraction.

above is similar conclusion to what we saw
   from 2-particle density matrix.

for small z, the leading term gives $\frac{N}{V} = \frac{g_s}{\lambda^3} z$

or $z = \left(\frac{N}{V} \frac{\lambda^3}{g_s} \right) = \frac{N}{Q_1}$ — same result we had classically

$\Rightarrow$ small z limit is the $\underline{\text{low density limit}}$ $n\lambda^3 \ll 1$

$$pV = N k_B T \left(1 \pm \frac{1}{2^{5/2} g_s} \frac{N}{V} \lambda^3 + \cdots \right) \qquad \underset{=}{\text{or high } T}$$

Sommerfeld model of electrons in a conductor
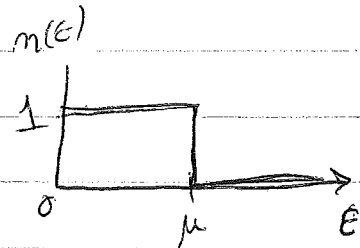
Fermi gas - high density / low temperature limit
"degenerate" fermi gas

Consider first $T \to 0$

$$\langle n(\epsilon) \rangle = \frac{1}{e^{\beta(\epsilon-\mu)}+1}$$

as $T \to 0$ $\quad e^{\beta(\epsilon-\mu)} \to \begin{cases} \infty & \epsilon > \mu \\ 0 & \epsilon < \mu \end{cases}$

$$\Rightarrow \langle n(\epsilon) \rangle \to \begin{cases} 0 & \epsilon > \mu \\ 1 & \epsilon < \mu \end{cases}$$

$\Rightarrow$ all states with $\epsilon < \mu$ are filled, all states with $\epsilon > \mu$ are empty. this is the $T=0$ ground state of the Fermi gas. We therefore see that $\mu(T=0)$ is the energy of the highest energy single particle state that is occupied in the ground state. One calls this energy the **Fermi-energy**

$$\epsilon_F \equiv \mu(T=0)$$

$T=0$

$$N = g_s \sum_{\vec{k}} 1 \qquad \text{count occupied states}$$

$\vec{k} \leftarrow$ such that $\frac{\hbar^2 k^2}{2m} < \epsilon_F$

$$\frac{\hbar^2 k_F^2}{2m} \equiv \epsilon_F$$

density $n = \dfrac{N}{V} = \dfrac{1}{V} g_s \sum_{\vec{k}} 1 = \displaystyle\int_0^{\epsilon_F} d\epsilon \, g(\epsilon)$

$$= \frac{2g_s}{\sqrt{\pi}} \left(\frac{2\pi m}{h^2}\right)^{3/2} \int_0^{\epsilon_F} d\epsilon \sqrt{\epsilon} = \frac{2g_s}{\sqrt{\pi}} \left(\frac{2\pi m}{h^2}\right)^{3/2} \frac{2}{3} \epsilon_F^{3/2}$$
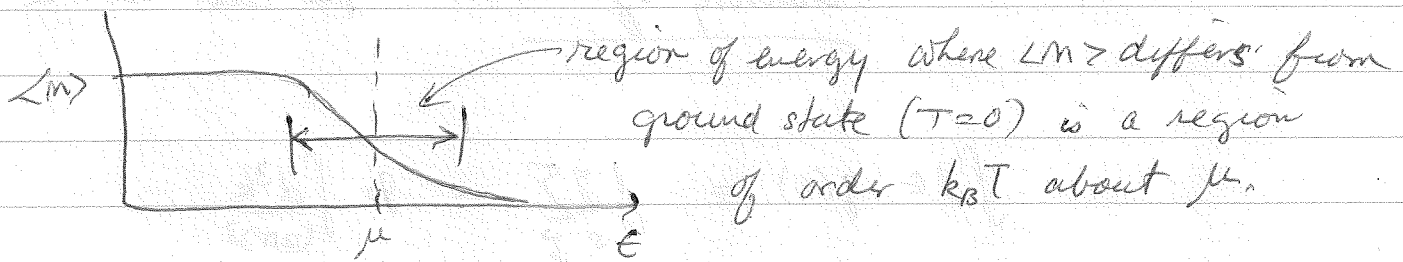
$$= \frac{2g_s}{\sqrt{\pi}} \left(\frac{2\pi m}{4\pi^2 \hbar^2}\right)^{3/2} \frac{2}{3} \epsilon_F^{3/2} = \frac{2g_s}{\sqrt{\pi} \, 8\pi^{3/2}} \frac{2}{3} \left(\frac{2m}{\hbar^2}\right)^{3/2} \epsilon_F^{3/2}$$

$$= \frac{g_s}{6\pi^2} \left(\frac{2m \, \epsilon_F}{\hbar^2}\right)^{3/2} \qquad \Rightarrow \boxed{\epsilon_F = \frac{\hbar^2}{2m} \left(\frac{6\pi^2 n}{g_s}\right)^{3/2}}$$

$\underbrace{}_{= k_F}$

Now at finite T

region of energy where $\langle n \rangle$ differs from ground state $(T=0)$ is a region of order $k_B T$ about μ.

So the $T \approx 0$ approx is good when $k_B T \ll \mu$

~~since $\mu(T) \approx \mu(0) = \epsilon_F$ we have~~

Using $\mu \approx \mu(0) = \epsilon_F$ we have

$$k_B T \ll \frac{\hbar^2}{2m}\left(\frac{6\pi^2 m}{g_s}\right)^{2/3} \Rightarrow \frac{2\pi m k_B T}{\hbar^2} \ll \frac{1}{4\pi}\left(\frac{6\pi^2 m}{g_s}\right)^{2/3}$$

$$\Rightarrow \lambda^2 \gg 4\pi \left(\frac{g_s}{6\pi^2 m}\right)^{2/3}$$

$$\Rightarrow n\lambda^3 \gg \frac{(4\pi)^{3/2}}{6\pi^2} g_s = \frac{4}{3\sqrt{\pi}} g_s$$

so this is equivalent to a low T or a high density limit $n\lambda^3 \gg 1$ — called the "degenerate" limit.

(just as the classical limit $z \approx n\lambda^3 \ll 1$ was a high T low density limit)

Fermi temperature $T_F \equiv \epsilon_F / k_B$. Degenerate limit is $T \ll T_F$

For electrons in a metal, $T_F \approx 10000\,^\circ K$.
So electrons in a metal are always in the degenerate limit.

$$g(\varepsilon) = C \sqrt{\varepsilon}$$

$$g(\varepsilon_F) = C \sqrt{\varepsilon_F}$$

$$m = \int_0^{\varepsilon_F} d\varepsilon \; g(\varepsilon) = \int_0^{\varepsilon_F} d\varepsilon \; C \; \varepsilon^{1/2} = \frac{2}{3} C \varepsilon_F^{3/2}$$

$$\Rightarrow \quad C = \frac{3}{2} \frac{m}{\varepsilon_F^{3/2}}$$

$$\boxed{g(\varepsilon_F) = \frac{3}{2} m \frac{\varepsilon_F^{1/2}}{\varepsilon_F^{3/2}} = \frac{3}{2} \frac{m}{\varepsilon_F}}$$

$$g(\varepsilon) = g(\varepsilon_F) \frac{g(\varepsilon)}{g(\varepsilon_F)} = \frac{3}{2} \frac{m}{\varepsilon_F} \sqrt{\frac{\varepsilon}{\varepsilon_F}}$$

Energy in the degenerate limit $T = 0$

$$\frac{E}{V} = \int_0^{\epsilon_F} d\epsilon \, g(\epsilon) \, \epsilon \qquad\qquad g(\epsilon) = C \sqrt{\epsilon}$$

$$\text{with} \quad C = \left(\frac{2\pi m}{h^2}\right)^{3/2} \frac{2g_s}{\sqrt{\pi}}$$

↗ density of states

$$n = \frac{N}{V} = \int_0^{\epsilon_F} d\epsilon \, g(\epsilon)$$

$$\Rightarrow \frac{E}{V} = C \int_0^{\epsilon_F} d\epsilon \, \epsilon^{3/2} = \frac{2}{5} C \, \epsilon_F^{5/2} \left.\right\}$$

$$n = \frac{N}{V} = C \int_0^{\epsilon_F} d\epsilon \, \epsilon^{1/2} = \frac{2}{3} C \, \epsilon_F^{3/2} \left.\right\} \Rightarrow \frac{E}{V} = \frac{3}{5} \frac{N}{V} \epsilon_F$$

$$\frac{E}{V} = \frac{3}{5} n \epsilon_F \qquad \text{or} \qquad \boxed{\frac{E}{N} = \frac{3}{5} \epsilon_F}$$
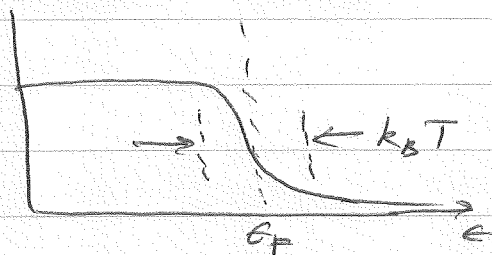
↑ energy per volume   ↑ energy per particle

Above gives $T=0$ results. To get behavior at low $T > 0$, or to get quantities such as $C_V = \left(\frac{\partial E}{\partial T}\right)_V$, we need to get the next order terms in a low temperature expansion.

In general we need to do integrals of the form

$$\int d\epsilon \, \frac{\tilde{\phi}(\epsilon)}{z^{-1} e^{\beta \epsilon} + 1} = \int d\epsilon \, \tilde{\phi}(\epsilon) \, n(\epsilon) \quad , \quad \tilde{\phi}(\epsilon) \text{ some function}$$

ex: to compute n, $\tilde{\phi}(\epsilon) = g(\epsilon)$; to compute $\frac{E}{V}$, $\tilde{\phi}(\epsilon) = g(\epsilon) \epsilon$

Simple estimate of C_V

When increase temperature to $k_B T$, the electrons near the fermi energy ϵ_F will increase their energy by an amount $\sim k_B T$. The number of such electrons ~~is roughly~~ per unit volume is roughly

$$g(\epsilon_F)(k_B T)$$

↑ density of states at ϵ_F

⌣ energy interval about ϵ_F of states which ~~increase~~ get excited

⇒ increase in energy per unit volume is

$$\Delta u \sim (g(\epsilon_F) k_B T)(k_B T) \sim g(\epsilon_F)(k_B T)^2$$

↑ # electrons excited

↑ excitation energy per excited electron

← for free electrons

$$\Rightarrow C_V = \frac{d \Delta u}{dT} \sim g(\epsilon_F) k_B^2 T = \frac{3}{2} \frac{m}{\epsilon_F} k_B^2 T = \frac{3}{2} m k_B \left(\frac{T}{T_F}\right)$$

The previous calculation gives the precise numerical coefficient