At $T=0$, $\mu = E_F$. As $T$ increases, does $\mu$ increase, decrease, or stay the same?

Note: $\mu$ must vary with temperature so that the density

$$M = \int_0^\infty dE \, m(E) f(E) \quad m(E) = \frac{1}{e^{E/k_BT} + 1}$$

stays constant.

Suppose $\mu$ stayed constant, $\mu = E_F$, as $T$ increased.

Because $\frac{dm}{dE}$ is symmetric about $\mu$, these two areas are equal.

The number of electrons that get removed from below $E_F$ is then $\sim g_0 (\mu - \frac{k_BT}{2}) k_BT$, while the number of electrons added above $E_F$ is $g_0 (\mu + \frac{k_BT}{2}) k_BT$. These two should be equal.

But since $g_0(E) \sim \sqrt{E}$ is an increasing function of $E$, there are more energy states within $k_BT$ above $E_F$ than there are below $E_F$, i.e. $g_0 (\mu + \frac{k_BT}{2}) k_BT > g_0 (\mu - \frac{k_BT}{2}) k_BT$.

The only way we can keep $\mu$ constant is for $\mu$ to decrease as $T$ increases. From Sommerfeld expansion:

$$\mu(T) = E_F - \frac{\pi^2}{3} (k_BT)^2 \frac{g'(E_F)}{g(E_F)} + \ldots$$
Pauli paramagnetism of electron gas

\[ \vec{s} = \frac{1}{\hbar} \vec{\sigma} \]

An electron has intrinsic spin \( \vec{s} \) with intrinsic magnetic moment \( \vec{\mu} = -\mu_B \vec{s} \) where \( \mu_B = \frac{10^{-4} \text{ Tesla}}{2mc} \) is Bohr magneton.

In an external magnetic field \( \vec{B} \), there is an interaction energy \( -\vec{\mu} \cdot \vec{B} = \mu_B \sigma B \) where \( \sigma = \pm 1 \) for spins parallel or antiparallel to \( \vec{B} \). The energy spectra for up and down electron spins becomes

\[ E_{\pm}(\vec{k}) = E_{0}(\vec{k}) \pm \mu_B B \]

where \( E_{0}(\vec{k}) \) is spectrum at \( \vec{B} = 0 \).

Since up and down electrons now have different energy spectra, we should treat them as two different populations of particles. They will be in equilibrium when their chemical potentials are equal, i.e., \( \mu_+ = \mu_- \). This will induce a net magnetization in the system.

To see this, consider free electrons at \( T = 0 \).
When $\mathbf{B} = 0$, $E_+(\mathbf{k}) = E_-(\mathbf{k})$; ground state occupations look as shown on the left. Equal numbers of $\uparrow$ and $\downarrow$ electrons $\mathcal{N}_+ = \mathcal{N}_-$.

When $\mathbf{B}$ is turned on, if there were no redistribution of electron spins, the situation would look like

Clearly the system can lower its energy by transferring $\uparrow$ electrons to $\downarrow$ electrons.

At equilibrium the system will look like

Again the two populations have the same max energy $E_F$. But there are now more $\downarrow$ electrons than $\uparrow$ electrons.

Magnetization

$$ \frac{M}{V} = -\mu_B \left( \mathcal{N}_+ - \mathcal{N}_- \right) > 0 $$

$\frac{M}{V}$ is parallel to $\mathbf{B} \implies$ paramagnetic effect
Let \( g_+(\varepsilon) \) be the density of states when \( B = 0 \).

When \( B > 0 \), the density of states for \( \uparrow \) and \( \downarrow \) electrons are

\[
\begin{align*}
g_+ (\varepsilon + \mu \pm \frac{1}{2} \varepsilon) & \Rightarrow g_+ (\varepsilon) = \frac{1}{2} g_+ (\varepsilon - \mu \pm \frac{1}{2} \varepsilon) \\
g_- (\varepsilon - \mu \pm \frac{1}{2} \varepsilon) & = \frac{1}{2} g_+ (\varepsilon) \\
g_- (\varepsilon) & = \frac{1}{2} g_+ (\varepsilon + \mu \pm \frac{1}{2} \varepsilon)
\end{align*}
\]

The density of \( \uparrow \) and \( \downarrow \) electrons is then

\[
m_\pm = \int_{E_{\text{min}}}^{E_{\text{max}}} d\varepsilon \ g_\pm (\varepsilon) f (\varepsilon, \mu (B)) \quad \varepsilon_{\text{min}} = \pm \mu B
\]

where

\[
f (\varepsilon, \mu (B)) = \frac{1}{e^{(\varepsilon - \mu (B)) / k_B T} + 1}
\]

\( \mu (B) \) is the chemical potential — it might depend on \( B \).

— it is same for \( \uparrow \) and \( \downarrow \).

We will consider only the case that

\[
\mu B \ll \mu (B) \approx E_F
\]

the spin interaction is small compared to \( E_F \).
First we will show that

\[ \mu(B) = \mu(B=0) \left[ 1 + O \left( \frac{MBB}{E_F} \right)^2 \right] \]

Since we will work in the \( MBB \ll E_F \) limit, we will then be able to ignore changes in \( \mu \) due to the finite \( B \) and just use \( \mu(B=0) \)

**Proof:** Consider the total density of electrons

\[ M = M^+ + M^- = \int \frac{d\varepsilon}{MBB} f(\varepsilon, \mu(B)) g^+(\varepsilon) - \int \frac{d\varepsilon}{MBB} f(\varepsilon, \mu(B)) g^-(\varepsilon) \]

\[ = \frac{1}{2} \int \frac{d\varepsilon}{MBB} f(\varepsilon, \mu(B)) g(\varepsilon - MBB) + \frac{1}{2} \int \frac{d\varepsilon}{MBB} f(\varepsilon, \mu(B)) g(\varepsilon + MBB) \]

\[ = \frac{1}{2} \int \frac{d\varepsilon}{MBB} g(\varepsilon) \left[ f(\varepsilon + MBB, \mu(B)) + f(\varepsilon - MBB, \mu(B)) \right] \]

Use the fact that \( f(\varepsilon, \mu) \) depends only on \( \varepsilon - \mu \)

\[ m = \frac{1}{2} \int \frac{d\varepsilon}{MBB} g(\varepsilon) \left[ f(\varepsilon, \mu(B) - MBB) + f(\varepsilon, \mu(B) + MBB) \right] \]

Expand \( f \) about \( \mu(B) \) for small \( MBB \)
\[ m = \frac{1}{2} \int \text{d} \epsilon \, g(\epsilon) \left[ f(\epsilon, \mu(B)) - \frac{d f}{d \mu} \mu_B + \frac{1}{2} \frac{d^2 f}{d \mu^2} (\mu_B^2)^2 + \cdots \right] + f(\epsilon, \mu(B)) + \frac{d f}{d \mu} \mu_B + \frac{1}{2} \frac{d^2 f}{d \mu^2} (\mu_B^2)^2 + \cdots \]

where derivatives above are evaluated at \( \mu = \mu(B) \),
the terms linear in \( B \) cancel!

\[ m = \int \text{d} \epsilon \, g(\epsilon) \left[ f(\epsilon, \mu(B)) + \frac{1}{2} \frac{d^2 f}{d \mu^2} (\mu_B^2)^2 + \cdots \right] \]

If we ignored the \((\mu_B^2)^2\) term the above
would be

\[ m = \int \text{d} \epsilon \, g(\epsilon) \, f(\epsilon, \mu(B)) \]

But this is just the same formula we use to
compute \( m \) at \( B=0 \). The magnetic field \( B \)
appears nowhere in the above, except via \( \mu(B) \).
Since the density is physically fixed by the sample
and cannot change as one varies \( B \), we would
conclude that

\[ \mu(B) = \mu(0) \text{ is independent of } B \]

Conclusion:
This depends on our having ignored the \((\mu_B^2)^2\) term,
so we can expect

\[ \mu(B) = \mu(0) + \frac{(\mu_B^2)^2}{\epsilon_F} \]

where \( \epsilon_F \) appears on dimensional grounds.
To see this is more explicitly, let's include the \((\mu B B)^2\) term and continue to calculate...

\[ m = \int d\varepsilon \ g(\varepsilon) \left[ f(\varepsilon, \mu B = 0) + \frac{1}{2} \frac{d^2 f}{d\mu^2} (\mu B B)^2 \right] \]

Write \(\mu B = \mu(B = 0) + 8\mu\) and expand in first term.

\[ m = \int d\varepsilon \ g(\varepsilon) \left[ f(\varepsilon, \mu(B = 0) + 8\mu) + \frac{1}{2} \frac{d^2 f}{d\mu^2} (\mu B B)^2 \right] \]

\[ = \int d\varepsilon \ g(\varepsilon) f(\varepsilon, \mu(B = 0)) \]

\[ + \int d\varepsilon \ g(\varepsilon) \frac{df}{d\mu} \bigg|_{\mu(B = 0)} 8\mu \]

\[ + \frac{1}{2} \int d\varepsilon \ g(\varepsilon) \frac{d^2 f}{d\mu^2} \bigg|_{\mu(B = 0)} (\mu B B)^2 \]

The first term is just the density when \(B = 0\), \(\sim m\), hence we get

\[ m = \int d\varepsilon \ g(\varepsilon) \frac{df}{d\mu} \bigg|_{\mu(B = 0)} 8\mu + \frac{1}{2} \int d\varepsilon \ g(\varepsilon) \frac{d^2 f}{d\mu^2} \bigg|_{\mu(B = 0)} (\mu B B)^2 \]

So the correction to \(\mu\) due to finite \(B\) is

\[ \delta\mu = -\frac{1}{2} \int d\varepsilon \ g(\varepsilon) \frac{d^2 f}{d\mu^2} \bigg|_{\mu(\mu B B)} (\mu B B)^2 \]

\[ \simeq -\frac{q(\varepsilon_F)}{2g(\varepsilon_F)} (\mu B B)^2 \]

\[ \int d\varepsilon \ g(\varepsilon) \frac{df}{d\mu} \bigg|_{\mu(B = 0)} \]
To see how big this is, consider the limit $T \to 0$
where $\mu(B=0) = E_F$, and $f$ is a step function.

\[ df = -\frac{df}{d\mu} = \delta(e-\mu) \]

\[ \frac{d^2f}{d\mu^2} = \frac{d^2f}{de^2} = \frac{d\delta(e-\mu)}{de} \]

So

\[ \int d\varepsilon g(\varepsilon) \frac{df}{d\mu} \bigg|_{\mu=\mu(B=0)} = g'(\mu(B)) \frac{\mu_B B^2}{2g(E_F)} \]

\[ \int d\varepsilon g(\varepsilon) \frac{d^2f}{d\mu^2} \bigg|_{\mu=\mu(B)} = g''(\mu(B)) \frac{\mu_B B^2}{g(E_F)} \]

\[ \bar{\lambda} = -\frac{1}{2} \frac{g'(\mu(B)) \mu_B B^2}{g(E_F)} \]

to lowest order, evaluate $g'(\mu(B))$ as $g'(E_F)$
The difference will only give higher order corrections

of $O(\mu_B)^4$

\[ \bar{\lambda} = -\frac{g'(E_F) \mu_B B^2}{2g(E_F)} \]

for free electrons with $g(\varepsilon) = \sqrt{\varepsilon}$ so

\[ g'(\varepsilon) = \frac{1}{2} \frac{c}{\sqrt{\varepsilon}} \]

we set

\[ \bar{\lambda} = -\frac{\mu_B B^2}{4E_F} \]

so

\[ \mu(B) = E_F \left( 1 - \frac{\mu_B B^2}{2E_F} \right) \]
Now we compute

\[ M = -\mu_B (m_+ - m_-) = \mu_B (m_+ - m_-) \]

\[ M = \mu_B \int \frac{d\varepsilon}{V} f(\varepsilon, \mu) \left[ g_-(\varepsilon) - g_+(\varepsilon) \right] \]

\[ = \mu_B \int \frac{d\varepsilon}{V} f(\varepsilon, \mu) \left[ \frac{1}{2} g(\varepsilon + \mu_B) - \frac{1}{2} g(\varepsilon - \mu_B) \right] \]

\[ = \frac{1}{2} \mu_B \int \frac{d\varepsilon}{V} g(\varepsilon) \left[ f(\varepsilon, \mu + \mu_B) - f(\varepsilon, \mu - \mu_B) \right] \text{ as before} \]

\[ \text{expand} \quad f(\varepsilon, \mu \pm \mu_B) = f(\varepsilon, \mu) \pm \frac{df}{d\mu} \mu_B \]

\[ M = \frac{1}{2} \mu_B \int \frac{d\varepsilon}{V} g(\varepsilon) \left[ \frac{1}{2} \frac{df}{d\mu} \mu_B \right] \]

\[ = \mu_B^2 B \int \frac{d\varepsilon}{V} g(\varepsilon) \left( -\frac{df}{d\varepsilon} \right) \quad \text{since} \quad \frac{df}{d\mu} = -\frac{df}{d\varepsilon} \]

To lowest order in temperature \(-\frac{df}{d\varepsilon} \sim 8(\varepsilon - \mu)\) with \(\mu = \varepsilon_F\)

\[ M = \frac{\mu_B^2 B g(\varepsilon_F)}{V} \]

could use Sommerfeld expansion to get corrections of order \(\left(\frac{k_B T}{\varepsilon_F}\right)^2\)

magnetic susceptibility \(X = \frac{2(M/V)}{B}\)

Pauli susceptibility \(\chi_p = \mu_B^2 g(\varepsilon_F)\) \(\propto\) density of states at \(\varepsilon_F\)

\(E_k = \frac{\hbar^2 k^2}{2m}\)

For free electron gas we earlier had \(g(\varepsilon_F) = \frac{3}{2} \frac{m}{\varepsilon_F}\)

\[ \Rightarrow \chi_p = \mu_B^2 \frac{3}{2} \frac{m}{\varepsilon_F} \]

\(\chi_p > 0 \Rightarrow\) paramagnetic
Compare this to classical result. Average magnetization of a single spin is
\[
<m> = \frac{\sum_{\pm} (-1)^{+ \pm} e^{-\beta M B}}{\sum_{\pm} e^{-\beta M B}} = \frac{e^{-\beta M B} + e^{\beta M B}}{e^{-\beta M B} + e^{\beta M B}}
\]
\[
<m> = \mu_B \tanh (\beta \mu_B B)
\]
\[
\frac{M}{V} = \frac{<m> N}{V} = \mu_B m \tanh (\beta \mu_B B)
\]
\[
\chi = \frac{d \left( \frac{M}{V} \right)}{dB}
\]

At low $T \to 0$, \( \tanh (\beta \mu_B) \to 1 \), \( \frac{M}{V} \to \mu_B m \), all spins aligned!

Compare to quantum case:
\[
\frac{M}{V} = \frac{\frac{3}{2} M}{\mu_B^2 B}
\]
Smaller than classical result by factor \( \frac{3}{2} \frac{\mu_B B}{E_F} \ll 1 \)

At high $T$ (\( \beta \gg 0 \)) \( \tanh (\beta \mu_B B) \to \beta \mu_B B \)
\[
\frac{M}{V} = \frac{\mu_B^2 B m}{k_B T}, \quad \chi = \frac{\mu_B^2 m}{k_B T} \sim \frac{1}{T}
\]
Compare to quantum case - at room temp, finite $T$ corrections remain negligible and still
\[
\chi_p = \mu_B^2 \frac{3}{2} \frac{m}{E_F} \quad \text{indeed of } T
\]
Smaller than classical by factor \( \frac{3}{2} \left( \frac{k_B T}{E_F} \right) \ll 1 \)