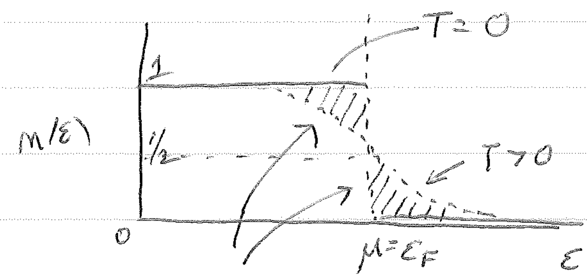
At $T=0$, $\mu = \varepsilon_F$. As T increases, does μ increase, decrease, or stay the same?
Note: μ must vary with temperature so that the density

$$n = \int_0^\infty d\varepsilon \, n(\varepsilon) g(\varepsilon) \qquad n(\varepsilon) = \frac{1}{z^{-1} e^{\beta\varepsilon} + 1}$$

stays constant.



Suppose μ stayed constant, $\mu = \varepsilon_F$, as T increased

because $\frac{\partial n}{\partial \varepsilon}$ is symmetric about μ, these two areas are equal.

The number of electrons that get removed from below ε_F is then $\sim g(\mu - \frac{k_BT}{2}) k_BT$, while the number of electrons added above ε_F is $g(\mu + \frac{k_BT}{2}) k_BT$. These two should be equal.
But since $g(\varepsilon) \sim \sqrt{\varepsilon}$ is an <u>increasing</u> function of ε there are more energy states within k_BT above ε_F than there are energy states within k_BT below ε_F, ie $g(\mu + \frac{k_BT}{2}) k_BT > g(\mu - \frac{k_BT}{2}) k_BT$

The only way we can keep n constant is for μ to <u>decrease</u> as T increases. From Sommerfeld expansion:

$$\mu(T) = \varepsilon_F - \frac{\pi^2}{6} (k_BT)^2 \frac{g'(\varepsilon_F)}{g(\varepsilon_F)} + \cdots$$

Pauli paramagnetism of electron gas

electron has intrinsic spin $\vec{\sigma}$ ($\underbrace{\vec{s} = \tfrac{1}{2}\hbar\vec{\sigma}}$) with intrinsic magnetic moment $\vec{\mu} = -\mu_B \vec{\sigma}$ $\quad \mu_B = \dfrac{|e|\hbar}{2mc}$ is Bohr magneton

In an external magnetic field $\vec{B}$, there is an interaction energy $-\vec{\mu}\cdot\vec{B} = \mu_B \sigma B$ where $\sigma = \pm 1$ for spins parallel and antiparallel to $\vec{B}$. The energy spectra for up and down electron spins becomes

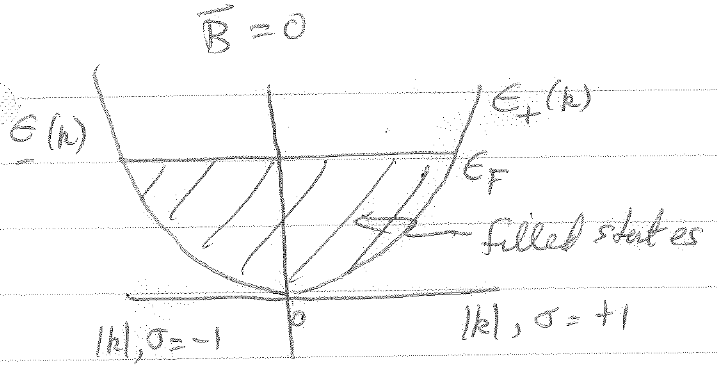
$$\epsilon_\pm(\vec{k}) = \epsilon(\vec{k}) \pm \mu_B B$$

where $\epsilon(\vec{k})$ is spectrum at $\vec{B} = 0$

Since $\uparrow$ and $\downarrow$ electrons now have different energy spectra, we should treat them as two different populations of particles $\Rightarrow$ ~~they~~ they will be in equilibrium when their chemical potentials are equal, ie $\mu_+ = \mu_-$

this will induce a net magnetization in the system.

To see this, consider free electrons at $T=0$

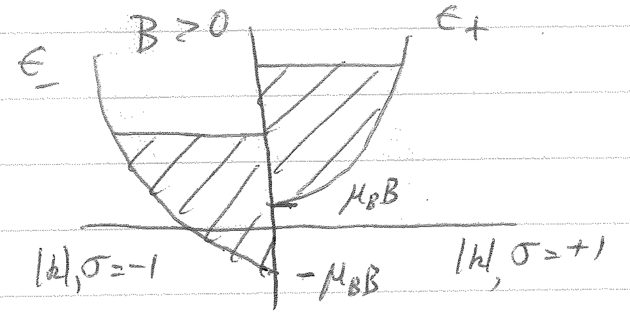


when $\vec{B} = 0$, $\epsilon_+(\vec{k}) = \epsilon_-(\vec{k})$

ground state occupations look as shown on the left. Equal numbers of $\uparrow$ and $\downarrow$ electrons

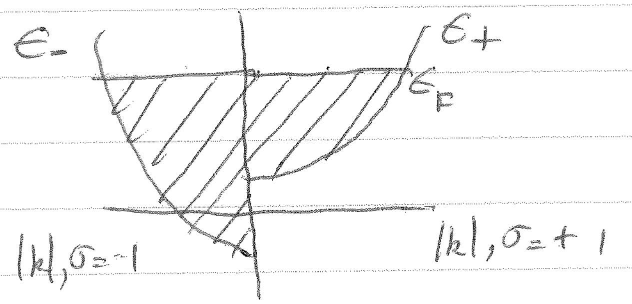
$n_+ = n_-$

When $\vec{B}$ is turned on, if there were no redistribution of electron spins, the situation would look like



Clearly the system can lower its energy by transferring $\uparrow$ electrons to $\downarrow$ electrons.

At equilibrium the system will look like



again the two populations have the same max energy ϵ_F. But there are now more $\downarrow$ electrons than $\uparrow$ electrons

magnetization $\frac{M}{V} = -\mu_B (n_+ - n_-) > 0$

$\frac{\vec{M}}{V}$ is parallel to $\vec{B}$ $\Rightarrow$ <u>paramagnetic effect</u>

The calculation

Let $g(\epsilon)$ be the density of states when $B=0$

When $B>0$, the density of states for ↑ and ↓ electrons are

$$g_+(\epsilon+\mu_B B) = \tfrac{1}{2} g(\epsilon) \Rightarrow g_+(\epsilon) = \tfrac{1}{2} g(\epsilon - \mu_B B)$$

$$g_-(\epsilon-\mu_B B) = \tfrac{1}{2} g(\epsilon) \qquad g_-(\epsilon) = \tfrac{1}{2} g(\epsilon+\mu_B B)$$

The density of ↑ and ↓ electrons is then

$$n_\pm = \int_{\varepsilon_{\pm min}}^{\infty} d\epsilon \; g_\pm(\epsilon) f(\epsilon, \mu(B)) \qquad \varepsilon_{\pm min} = \pm \mu_B B$$

where $f(\epsilon, \mu(B)) = \dfrac{1}{e^{(\epsilon-\mu(B))/k_BT}+1}$

$\mu(B)$ is the chemical potential – it might depend on B

– it is same for ↑ and ↓

We will consider only the case that

$$\mu_B B \ll \mu(B) \approx \varepsilon_F$$

ie spin interaction is small compared to ϵ_F

First we will show that

① $\mu(B) \simeq \mu(B=0)\left[1 + O\left(\frac{\mu_B B}{\varepsilon_F}\right)^2\right]$

Since we will work in the $\mu_B B \ll \varepsilon_F$ limit, we will then be able to ignore changes in μ due to the finite B and just use $\mu(B=0)$

proof: Consider the total density of electrons

$$n = n_+ + n_- = \int_{\mu_B B}^{\infty} d\varepsilon\, f(\varepsilon,\mu(B))\, g_+(\varepsilon) + \int_{-\mu_B B}^{\infty} d\varepsilon\, f(\varepsilon,\mu(B))\, g_-(\varepsilon)$$

$$= \frac{1}{2}\int_{\mu_B B}^{\infty} d\varepsilon\, f(\varepsilon,\mu(B))\, g(\varepsilon-\mu_B B) + \frac{1}{2}\int_{-\mu_B B}^{\infty} d\varepsilon\, f(\varepsilon,\mu(B))\, g(\varepsilon+\mu_B B)$$

shift integration variable $\varepsilon - \mu_B B \to \varepsilon$ shift integration variable $\varepsilon + \mu_B B \to \varepsilon$

$$= \frac{1}{2}\int_{0}^{\infty} d\varepsilon\, g(\varepsilon)\left[f(\varepsilon+\mu_B B, \mu(B)) + f(\varepsilon-\mu_B B, \mu(B))\right]$$

use the fact that $f(\varepsilon,\mu)$ depends only on $\varepsilon-\mu$

$$n = \frac{1}{2}\int_{0}^{\infty} d\varepsilon\, g(\varepsilon)\left[f(\varepsilon, \mu(B)-\mu_B B) + f(\varepsilon, \mu(B)+\mu_B B)\right]$$

expand f about $\mu(B)$ for small $\mu_B B$

$$n \simeq \frac{1}{2}\int d\epsilon\, g(\epsilon)\Big[f(\epsilon,\mu(B)) - \frac{df}{d\mu}\mu_B B + \frac{1}{2}\frac{d^2 f}{d\mu^2}(\mu_B B)^2 + \cdots$$

$$+ f(\epsilon,\mu(B)) + \frac{df}{d\mu}\mu_B B + \frac{1}{2}\frac{d^2 f}{d\mu^2}(\mu_B B)^2 + \cdots\Big]$$

where derivatives above are evaluated at $\mu = \mu(B)$. The terms linear in B cancel!

$$n \simeq \int d\epsilon\, g(\epsilon)\Big[f(\epsilon,\mu(B)) + \frac{1}{2}\frac{d^2 f}{d\mu^2}(\mu_B B)^2 + \cdots\Big]$$

If we ignored the $(\mu_B B)^2$ term the above would be

$$n = \int d\epsilon\, g(\epsilon)\, f(\epsilon,\mu(B))$$

But this is just the same formula we use to compute n at $B=0$! The magnetic field B appears nowhere in the above, except via $\mu(B)$. Since the density is physically fixed by the sample and cannot change as one varies B, we would conclude that

$$\mu(B) = \mu(0) \text{ is independent of } B!$$

This conclusion depends on our having ignored the $(\mu_B B)^2$ term, so we can expect

$$\mu(B) \simeq \mu(0) + \frac{(\mu_B B)^2}{\epsilon_F}$$

where $\frac{1}{\epsilon_F}$ appears on dimensional grounds.

To see this is so more explicitly, lets include the $(\mu_B B)^2$ term and continue to calculate ...

$$n = \int d\epsilon\, g(\epsilon) \left[f(\epsilon, \mu(B)) + \frac{1}{2} \frac{d^2 f}{d\mu^2} (\mu_B B)^2 \right]$$

write $\mu(B) = \mu(B=0) + \delta\mu$ and expand in first term

$$n = \int d\epsilon\, g(\epsilon) \left[f(\epsilon, \mu(B=0) + \delta\mu) + \frac{1}{2} \frac{d^2 f}{d\mu^2} (\mu_B B)^2 \right]$$

$$= \int d\epsilon\, g(\epsilon)\, f(\epsilon, \mu(B=0))$$

$$+ \int d\epsilon\, g(\epsilon) \left.\frac{df}{d\mu}\right|_{\mu=\mu(B=0)} \delta\mu$$

$$+ \frac{1}{2} \int d\epsilon\, g(\epsilon) \left.\frac{d^2 f}{d\mu^2}\right|_{\mu=\mu(B)} (\mu_B B)^2$$

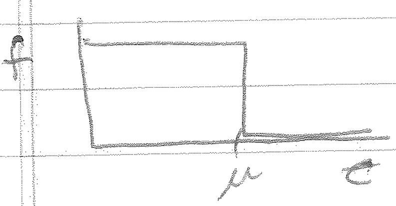
The first term is just the density when $B=0$, i.e. n. Hence we get

$$0 = \int d\epsilon\, g(\epsilon) \left.\frac{df}{d\mu}\right|_{\mu=\mu(B=0)} \delta\mu + \frac{1}{2} \int d\epsilon\, g(\epsilon) \left.\frac{d^2 f}{d\mu^2}\right|_{\mu=\mu(B)} (\mu_B B)^2$$

So the correction to μ due to finite B is

$$\delta\mu = \frac{-\frac{1}{2} \int d\epsilon\, g(\epsilon) \left.\frac{d^2 f}{d\mu^2}\right|_{\mu=\mu(B)} (\mu_B B)^2}{\int d\epsilon\, g(\epsilon) \left.\frac{df}{d\mu}\right|_{\mu=\mu(B=0)}} \approx -\frac{g'(\varepsilon_F)}{2 g(\varepsilon_F)} (\mu_B B)^2$$

To see how big this is, consider the limit $T \to 0$ where $\mu(B=0) = \epsilon_F$, and f is a step function



$$\frac{df}{d\mu} = -\frac{df}{d\epsilon} = \delta(\epsilon - \mu)$$

$$\frac{d^2f}{d\mu^2} = \frac{d^2f}{d\epsilon^2} = -\frac{d\,\delta(\epsilon-\mu)}{d\epsilon}$$

so $\displaystyle\int d\epsilon\, g(\epsilon) \left.\frac{df}{d\mu}\right|_{\mu = \mu(B=0)} = g(\mu(B=0)) = g(\epsilon_F)$

$$\int d\epsilon\, g(\epsilon) \left.\frac{d^2f}{d\mu^2}\right|_{\mu=\mu(B)} (\mu_B B)^2 \simeq g'(\mu(B))\,(\mu_B B)^2$$

$$\delta\mu = -\frac{1}{2}\,\frac{g'(\mu(B))\,(\mu_B B)^2}{g(\epsilon_F)}$$

to lowest order, evaluate $g'(\mu(B))$ as $g'(\epsilon_F)$
The difference will only give higher order corrections of $O(\mu_B B)^4$

$$\delta\mu = -\frac{g'(\epsilon_F)(\mu_B B)^2}{2g(\epsilon_F)}$$

for free electrons with $g(\epsilon) = c\sqrt{\epsilon}$ so
$g'(\epsilon) = \frac{1}{2}\frac{c}{\sqrt{\epsilon}}$ we get

$$\boxed{\delta\mu = -\frac{(\mu_B B)^2}{4\epsilon_F}} \quad \text{so} \quad \mu(B) = \epsilon_F\left(1 - \left(\frac{\mu_B B}{2\epsilon_F}\right)^2\right)$$

Now we compute

(2) Magnetization $\frac{M}{V} = -\mu_B(n_+ - n_-) = \mu_B(n_- - n_+)$

$$\frac{M}{V} = \mu_B \int d\epsilon\, f(\epsilon,\mu)\left[g_-(\epsilon) - g_+(\epsilon)\right]$$

$$= \mu_B \int d\epsilon\, f(\epsilon,\mu)\left[\tfrac{1}{2} g(\epsilon+\mu_B B) - \tfrac{1}{2} g(\epsilon - \mu_B B)\right]$$

$$= \tfrac{1}{2}\mu_B \int d\epsilon\, g(\epsilon)\left[f(\epsilon,\mu+\mu_B B) - f(\epsilon,\mu-\mu_B B)\right] \quad \text{as before}$$

expand $f(\epsilon,\mu \pm \mu_B B) = f(\epsilon,\mu) \pm \frac{df}{d\mu}\mu_B B$

$$\frac{M}{V} = \tfrac{1}{2}\mu_B \int d\epsilon\, g(\epsilon)\left[2\frac{df}{d\mu}\mu_B B\right]$$

$$= \mu_B^2 B \int_{-\infty}^{\infty} d\epsilon\, g(\epsilon)\left(-\frac{\partial f}{\partial \epsilon}\right) \qquad \text{since } \frac{\partial f}{\partial \mu} = -\frac{\partial f}{\partial \epsilon}$$

To lowest order in temperature $-\frac{\partial f}{\partial \epsilon} \approx \delta(\epsilon - \mu)$ with $\mu = \epsilon_F$

$$\boxed{\frac{M}{V} = \mu_B^2 B\, g(\epsilon_F)}$$

could use Sommerfeld expansion to get corrections of order $\left(\frac{k_B T}{\epsilon_F}\right)^2$

magnetic susceptibility $\chi = \frac{\partial(M/V)}{\partial B}$

Pauli susceptibility $\boxed{\chi_P = \mu_B^2 g(\epsilon_F)}$ ~ density of states at ϵ_F

$\epsilon_k = \hbar^2 k^2/2m$

For free electron gas we earlier had $g(\epsilon_F) = \frac{3}{2}\frac{n}{\epsilon_F}$

$$\Rightarrow \boxed{\chi_P = \mu_B^2 \frac{3}{2}\frac{n}{\epsilon_F}} \qquad \chi_P > 0 \Rightarrow \text{paramagnetic}$$

Compare this to classical result. Average magnetization of a single spin is

$$\langle m \rangle = (-\mu_B)\left[\frac{e^{-\beta\mu_B B}(+1) + e^{+\beta\mu_B B}(-1)}{e^{\beta\mu_B B} + e^{-\beta\mu_B B}}\right]$$

$$\langle m \rangle = \mu_B \tanh(\beta\mu_B B)$$

$$\frac{M}{V} = \langle m \rangle \frac{N}{V} = \mu_B n \tanh(\beta\mu_B B)$$

$$\chi = \frac{d(M/V)}{dB}$$

at low $T \to 0$, $\tanh(\beta\mu B) \to 1$, $\frac{M}{V} \to \mu_B n$

all spins alligned!

Compare to quantum cas:

$$\frac{M}{V} = \frac{3}{2}\frac{n}{\epsilon_F}\mu_B^2 B$$

Smaller than classical result by factor $\frac{3}{2}\frac{\mu_B B}{\epsilon_F} \ll 1$

at high T $(\beta \to 0)$ $\tanh(\beta\mu_B B) \longrightarrow \beta\mu_B B$

$$\frac{M}{V} = \frac{\mu_B^2 B n}{k_B T}, \qquad \chi = \frac{\mu_B^2 n}{k_B T} \sim \frac{1}{T}$$

Compare to quantum case - at room temp finite T corrections remain negligible ad still

$$\chi_p = \mu_B^2 \frac{3}{2}\frac{n}{\epsilon_F} \quad \text{indep of } T$$

smaller than classical by factor $\frac{3}{2}\left(\frac{k_B T}{\epsilon_F}\right) \ll 1$