Ideal Bose Gas

Bose-Einstein Condensation

Bose occupation function

\[ n(\epsilon) = \frac{1}{\epsilon - \beta \epsilon - 1} \]

We had for the density of an ideal (non-interacting) Bose gas

\[ \frac{N}{V} = \frac{1}{V} \sum_k \frac{1}{\epsilon_k - \beta \epsilon_k} = \frac{1}{(2\pi)^3} \int_0^\infty dk \frac{4\pi k^2}{Z} \frac{1}{\epsilon_k - \beta \hbar^2 k^2 / 2m - 1} \]

recall, we need \( Z \leq 1 \) for the occupation number

at \( \epsilon_k(0) = 0 \) to remain positive \( M(0) > 0 \)

\[ M(0) = \frac{1}{Z - 1} = \frac{Z}{1 - Z} \Rightarrow Z \leq 1 \Rightarrow Z = e^{\beta \mu/2} \Rightarrow M \leq 0 \]

Substitute variables \( y = \frac{\beta \hbar^2 k^2}{2m} \Rightarrow k = \sqrt{\frac{2my}{\beta \hbar^2}} \)

\[ dk = \frac{2m}{\beta \hbar^2} \frac{dy}{y} \]

\[ \Rightarrow \frac{N}{V} = \left( \frac{2m}{\beta \hbar^2} \right)^{3/2} \frac{4\pi}{(2\pi)^3} \frac{1}{2} \int_0^\infty dy \frac{y^{1/2}}{Z - e^y - 1} \]

\[ \frac{N}{V} = \frac{1}{\lambda^3} g_{3/2}(Z) \]

where \( \lambda = \left( \frac{\hbar^2}{2\pi m k_B T} \right)^{1/2} \) thermal wavelength

\[ g_{3/2}(Z) = \frac{2}{\sqrt{\pi}} \int_0^\infty dy \frac{y^{1/2}}{Z - e^y - 1} \]
Consider the function
\[
G_{3/2}(z) = \frac{2}{\pi} \int_0^\infty \frac{y^{1/2}}{z^{3/2} - e^y - 1} \, dy = z \sum_{n=1}^{\infty} \frac{z^n}{z^{3/2} n^{3/2}}
\]

\(G_{3/2}(z)\) is a monotonic increasing function of \(z\) for \(z \leq 1\).

As \(z \to 1\), \(G_{3/2}(z)\) approaches a finite constant.

\[
G_{3/2}(1) = 1 + \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} + \cdots \leq \zeta(3/2) \approx 2.2012.
\]

We can see that \(G_{3/2}(1)\) is finite as follows:

\[
G_{3/2}(1) = \frac{2}{\pi} \int_0^\infty \frac{y^{1/2}}{e^y - 1} \, dy \quad \text{as } y \to \infty \text{ the integral converges, integral is largest at small } y
\]

(recall small \(y\) corresponds to low energy where mc\(e\) is largest)

For small \(y\) we can approximate \(\frac{1}{e^y - 1} \approx \frac{1}{y}\).

\[
\int_0^y \frac{y^{1/2}}{e^{y/2} - 1} \, dy \approx \int_0^y \frac{1}{y^{1/2}} \, dy = 2 y^{1/2} \bigg|_0^y
\]

So we see the integral also converges at its lower limit \(y \to 0\).
So we conclude

\[ n = \frac{N}{V} = \frac{9^{3/2}(1)}{2^3} \leq \frac{9^{3/2}(1)}{2^3} = \frac{2.612}{2.612} \left( \frac{2\pi m k_B T}{\hbar^2} \right)^{3/2} \]

But we now have a contradiction!

For a system with fixed density of bosons \( n \), as \( T \) decreases we will eventually get to a temperature below which the above inequality is violated.

This temperature is

\[ T_0 = \left( \frac{m}{2.612} \right)^{2/3} \frac{\hbar^2}{2\pi m k_B} \]

**Solution to the paradox:**

When we made the approx \( \frac{1}{V} \sum_k \to \frac{1}{(2\pi)^3} \int dk \frac{4\pi k^2}{(2\pi)^3} \)

we gave a weight \( \frac{4\pi k^2}{(2\pi)^3} \) to states with wavevector \( |\mathbf{k}| \).

This gives zero weight to the state \( |\mathbf{k}| = 0 \), i.e. to the ground state. But as \( T \) decreases, more and more bosons will occupy the ground state, as it has the lowest energy. Thus, when we approx the sum by an integral, we should treat the ground state separately.

\[ \frac{1}{V} \sum_k n(E(k)) \approx \frac{n(0)}{V} + \frac{1}{(2\pi)^3} \int dk \frac{4\pi k^2}{(2\pi)^3} n(E(k)) \]

Ground state with occupation \( n(0) \).

This term is important when \( n(0)/V \) stays finite as \( V \to \infty \), i.e. a macroscopic fraction of bosons occupy the ground state.
Then we get

\[ M = \frac{N}{V} = \frac{n(0)}{V} + \frac{g_{3/2}(z)}{\lambda^3} \]

\[ M = M_0 + \frac{g_{3/2}(z)}{\lambda^3} \quad \text{where } M_0 = \frac{n(0)}{V} \text{ density of bosons in ground state} \]

For a system with fixed \( M \), at higher \( T \) one can always choose \( z \) so that \( M = \frac{g_{3/2}(z)}{\lambda^3} \) and \( M_0 = 0 \).

But when \( T < T_c \) it is necessary to have \( M_0 > 0 \).

Using \( n(0) = \frac{z}{1-z} \) we can write above as

\[ M = \frac{z}{1-z} \frac{1}{V} + \frac{g_{3/2}(z)}{\lambda^3} \]

For \( T > T_c \), we will have a solution to the above for some fixed \( z < 1 \). In thermodynamic limit \( V \to \infty \), the first term will then vanish, i.e. the density of bosons in the ground state vanishes.

As \( V \to \infty \) as \( T \to T_c \), \( z \to 1 \) and the first term \( \frac{z}{1-z} \frac{1}{V} \) stays finite to give the additional needed density at \( T < T_c \):

\[ \frac{z}{1-z} \frac{1}{V} = M_0 = m - \frac{g_{3/2}(1)}{\lambda^3} \]
To define the Bose-Einstein transition temperature below which the system develops a finite density of particles in the ground state $N_0$.

$N_0$ is also called the condensate density.

The particles in the ground state are called the condensate.

$Z(T) \rightarrow 1^2$ as $T \rightarrow T_c$

$Z(T) = 1^2$ for $T \leq T_c$

$\mu(T) \rightarrow 0$

$\mu(T) = 0$

For $T \leq T_c$

$$N_0(T) = m - \frac{g \sqrt{z}}{1!} \left( \frac{2\pi m k_B T}{\hbar^2} \right)^{3/2}$$

$$N_0(T) = m \left( 1 - \left( \frac{T}{T_c} \right)^{3/2} \right)$$

condensate density vanishes continuously as $T \rightarrow T_c$ from below.

At $T = 0$, all bosons are in condensate.

At $T > T_c$, all bosons are in the "normal state".

At $0 < T < T_c$, a macroscopic fraction of bosons are in the condensate, while the remaining fraction are in the normal state. This state is called the "mixed state".
\[ \Phi = \frac{1}{V} \ln \mathcal{Z} = -\frac{1}{V} \sum_{k} \ln \left( 1 - Z e^{-\beta \mathcal{E}(k)} \right) \]

\[ = -\frac{1}{V} \ln \left( 1 - Z \right) - \frac{4 \pi}{(2\pi)^{3}} \int_{0}^{\infty} \frac{d k}{k^{2}} \ln \left( 1 - Z e^{-\beta \frac{k^{2} m}{2} \mathcal{H}_{0}} \right) \]

\[
\begin{align*}
\text{\(k=0\) ground state} & \quad \text{\(\uparrow\) all other \(|k|>0\) states} \\
\mathcal{Z} & = \left( \frac{k^{2}}{2\pi m k_{B} T} \right)^{\frac{3}{2}} \\
\end{align*}
\]

where \( g_{\frac{3}{2}} (Z) \equiv \frac{1}{\Gamma (\frac{3}{2})} \int_{0}^{\infty} \frac{dy}{y^{\frac{3}{2}}} e^{-y} \) as derived when we began our discussion of quantum gases.

Also recall the number of bosons occupying the ground state is

\[ n(0) = \frac{1}{Z^{-1} e^{\beta \mathcal{E}(0)} - 1} = \frac{1}{Z^{-1} - 1} = \frac{Z}{1 - Z} \]

So

\[ n(0) + 1 = \frac{Z}{1 - Z} + 1 = \frac{1}{1 - Z} \]

\[ \Phi = \frac{1}{V} \ln \left( n(0) + 1 \right) + \frac{g_{\frac{3}{2}} (Z)}{2^3} \]

In the thermodynamic limit of \( V \to \infty \), the first term always vanishes as \( n(0) \ll N = nV \) and

\[ \lim_{V \to \infty} \left[ \frac{\ln(nV)}{V} \right] = 0 \]

So the condensate does not contribute to the pressure. This is not surprising as particles in the condensate have \( k = 0 \) and hence carry no momentum. In the kinetic theory of gases, one sees that pressure arises from particles with finite momentum \( |\mathbf{p}| > 0 \) hitting the walls of the container.
So \( \frac{\Phi}{k_B T} = \frac{g_{3/2}(z)}{z^3} = \Phi_{3/2}(z) \left( \frac{2\pi m k_B T}{\hbar^2} \right)^{3/2} \)

\[ \Phi = \Phi_{3/2}(z(T)) \left( \frac{2\pi m k_B T}{\hbar^2} \right)^{3/2} \]

Equation of state

For a system of fixed density \( m \), \( z \) must be chosen to be a function of \( T \) that gives the desired density \( m \).

\[ g_{3/2}(z=1) = \Phi_{3/2}(1/2) = 1.342 \]

is finite

In the thermodynamic limit of \( V \to \infty \), \( z=1 \) for \( T \leq T_c(m) \)

\[ \Rightarrow \Phi = \Phi_{3/2}(1) \left( \frac{2\pi m k_B T}{\hbar^2} \right)^{3/2} \] for \( T \leq T_c \)

Critical temperature

depends on the system's fixed density

Note: for \( T \leq T_c \), the pressure \( \rho \propto T^{5/2} \) is independent of the system density!

\[ p \propto T \text{ curves at constant density } m \]

\[ T_c(m) \sim m^{2/3} \]

Recall \( T_c(m) \sim m^{2/3} \frac{2\pi m k_B}{2\pi m k_B} \)

\[ T_c(m) = (\frac{m}{2^{1/6}})^{2/3} \frac{\hbar^2}{2\pi m k_B} \]
Define \( M_c(T) = \frac{2.6 \times 10^{-12} (2\pi m k_B T)^{3/2}}{\hbar^2} \) inverse of \( T_c(m) \)

\( M_c(T) \) is the critical density at a given \( T \)
- a system with \( \rho > M_c(T) \) will be in a
  - bose condensed mixed state at temperature \( T \).

Phase diagram in \( p-T \) plane

Can also consider the transition in terms of \( p \) and \( \nu = \frac{V}{N} = \frac{1}{m} \) for various fixed \( T \).

At the transition \( p \propto T_c(m)^{5/2} \) \( \rightarrow T_c(m) \propto m^{2/3} \)

\[ \Rightarrow \text{at the transition} \quad p \propto (m^{2/3})^{5/2} = m^{5/3} = \nu^{5/3} \]

below the transition \( p \) is independent of \( \rho \) density and hence independent of \( V \).

For fixed \( T \), the transition occurs when density \( \rho \) exceeds \( M_c(T) \), or when \( \nu \) drops below \( \nu_c(T) = \frac{1}{M_c(T)} \)

\[ \nu_c(T) \propto T^{-3/2} \]
curves of $p$ vs $v$ at constant $T$

Thermodynamic functions

Earlier we found $\frac{E}{N} = \frac{3}{2} p$

$$\Rightarrow \frac{E}{N} = \frac{3}{2} \rho \frac{V}{N} = \frac{3}{2} \rho v = \frac{3}{2} \frac{k_B T v}{\lambda^3} \rho \frac{g_{\frac{3}{2}}(z)}{z} \uparrow$$

$z=1$ in mixed state
$z<1$ in normal state

In above we regard $\frac{E}{N}$ as a function of either $v$ or $z$. That is, we either determine $v$ for a given $z, T$ or we determine $z$ needed for a given $v, T$ (Recall $z = e^\mu$, $v = \frac{V}{N}$ and $N$ and $\mu$ are conjugate variables)

Specific heat

$$\frac{C_v}{Nk_B} = \left. \left( \frac{\partial (E/N)}{\partial T} \right) \right|_{v,N} = \frac{3}{2} v \left( \frac{d}{dT} \frac{T}{\lambda^3} \right) g_{\frac{3}{2}}(z) + \frac{T}{\lambda^3} \frac{\partial g_{\frac{3}{2}}(z)}{\partial z} \frac{dz}{dT}$$
For $T \leq T_c$, $\zeta = 1$ so $\frac{d\zeta}{dT} = 0$ and only 1st term remains

$$\frac{I}{\lambda^3} = T^{5/2} \quad \text{so} \quad \frac{d}{dT} \left( \frac{I}{\lambda^3} \right) = \frac{5}{2} \left( \frac{I}{\lambda^3} \right) \frac{1}{T} = \frac{5}{2} \frac{1}{\lambda^3}$$

$\zeta = 1$ here for all $T \leq T_c$

$$\Rightarrow \frac{C_V}{N k_B} = \frac{3}{2} \nu \left( \frac{5}{2} \frac{1}{\lambda^3} \right) g_{3/2}(1) = \frac{15}{4} g_{3/2}(1) \frac{\nu}{\lambda^3}$$

$$= \frac{15}{4} g_{3/2}(1) \nu \left( \frac{2 \pi m k_B T}{\hbar^2} \right)^{3/2}$$

Note, at $T_c$, $\frac{m}{\lambda_c^3} = g_{3/2}(1)$ and $\nu = \frac{1}{m}$

$$\frac{C_V(T_c)}{N k_B} = \frac{15}{4} \frac{g_{3/2}(1)}{4} = \frac{15}{2} \times 1.341 = 19.25$$

this is

the classical ideal gas value of $\frac{3}{2}$

So

$$\left( \frac{C_V}{N k_B} \right) = 1.925 \left( \frac{T}{T_c} \right)^{3/2} \quad T \leq T_c$$

For $T > T_c$, $\zeta$ varies with $T$ and we need to evaluate the 2nd term as well

1st term gives

$$\frac{15}{4} g_{3/2}(\zeta(T)) \frac{\nu}{\lambda^3}$$

2nd term: from Pathria Appendix B Eq(10),

$$\frac{d}{dz} \left[ g_{\nu}(z) \right] = g_{\nu-1}(z)$$

$$\Rightarrow \frac{d g_{3/2}}{dz} \frac{dT}{dz} = g_{3/2} \frac{1}{2} \frac{dT}{dz}$$
To find \( \frac{1}{3} \frac{d^2}{dT} \) consider our earlier result for the density when \( T > T_c \):

\[ n = \frac{g_{3/2}(z)}{a^3} \] - determines \( z(T) \) for fixed \( n \)

for \( n \) fixed \( \Rightarrow \)

\[ 0 = \frac{d}{dT} \left( \frac{1}{a^3} \right) g_{3/2} + \frac{1}{a^3} \frac{d}{dz} g_{3/2} \frac{dz}{dT} \]

\[ 0 = \frac{3}{2} \frac{1}{a^3} \int g_{3/2} + \frac{1}{a^3} \int g_{1/2} \frac{1}{a} \frac{dz}{dT} \]

\( \Rightarrow \)

\[ \frac{d}{dT} \int g_{3/2} = -\frac{3}{2} \frac{g_{3/2}}{g_{1/2}} \frac{1}{T} \]

\[ \frac{C_v}{N k_B} = \frac{15}{4} \frac{g_{3/2}(z)}{a^3} - \frac{9}{4} \frac{g_{1/2}(z)}{a^3} \left( \frac{3}{2} \right) \frac{g_{3/2}(z)}{g_{1/2}(z)} \frac{1}{T} \]

\[ \text{use} \quad n = \frac{1}{V} = \frac{g_{3/2}(z)}{a^3} \Rightarrow \frac{V}{a^3} = \frac{g_{3/2}(z)}{T} \]

\[ \frac{C_v}{N k_B} = \frac{15}{4} \frac{g_{3/2}(z)}{g_{3/2}(z)} - \frac{9}{4} \frac{g_{1/2}(z)}{g_{3/2}(z)} \quad T > T_c \]

Note \( g_{1/2}(1) = \frac{\phi}{\epsilon - \phi} \frac{1}{\epsilon^{1/2}} \rightarrow \infty \)

So as \( T \rightarrow T_c^+ \) from above, \( \text{and} \) \( z \rightarrow 1 \)

\[ \frac{C_v(T_c^+)}{N k_B} = \frac{15}{4} \frac{g_{3/2}(1)}{g_{3/2}(1)} - \frac{9}{4} \frac{g_{3/2}(1)}{\infty} = \frac{15}{4} 1.841 = 1.925 \]

\[ \Rightarrow \]

\( C_v \) is continuous at \( T_c \)
Finally we want to show that although $C_V$ is continuous at $T_c$, $\frac{dC_V}{dT}$ is discontinuous.

For $T \leq T_c$,
\[
C_V = 1.925 \left( \frac{T}{T_c} \right)^{3/2} \frac{N_k}{N_k_B}
\]

\[
\frac{d}{dT} \left( \frac{C_V}{N_k_B} \right) = \frac{3}{2} \left( 1.925 \right) \left( \frac{T}{T_c} \right)^{1/2} \frac{1}{T_c} = 2.89 \left( \frac{T}{T_c} \right)^{1/2} \frac{1}{T_c}
\]

so slope at $T_c^-$(just below $T_c$)

\[
\frac{d}{dT} \left( \frac{C_V}{N_k_B} \right) = \frac{2.89}{T_c} \quad T = T_c^{-}
\]

For $T > T_c$
\[
C_V = \frac{15}{4} g_{3/2}(z) - \frac{9}{4} g_{1/2}(z)
\]

\[
\frac{d}{dT} \left( \frac{C_V}{N_k_B} \right) = \frac{15}{4} g_{3/2} \frac{dg_{3/2}}{dz} \frac{dz}{dT} - g_{3/2} \frac{dg_{3/2}}{dT} \frac{dz}{dT}
\]

\[
= -\frac{9}{4} g_{1/2} \frac{dg_{1/2}}{dz} \frac{dz}{dT} - g_{1/2} \frac{dg_{1/2}}{dT} \frac{dz}{dT}
\]

\[
= \frac{1}{2} \frac{d^2z}{dT^2} \left\{ 15 \left( \frac{g_{3/2} - g_{5/2} g_{1/2}}{g_{3/2}^2} \right) - \frac{9}{4} \left( \frac{g_{1/2}^2 - g_{3/2}^2 g_{1/2}}{g_{1/2}^2} \right) \right\}
\]

Use \[
\frac{1}{2} \frac{d^2z}{dT^2} = -\frac{3}{2} \frac{g_{3/2}}{g_{1/2}} \frac{1}{T} \quad \text{as found earlier}
\]
\[
\frac{d}{dT} \left( \frac{C_V}{Nk_B} \right) = -\frac{3}{8T} \left( \frac{\theta^{3/2}}{\theta^{1/2}} \right) \left\{ 15 \left( 1 - \frac{\theta^{3/2}}{\theta^{5/2}} \right) - 9 \left( 1 - \frac{\theta^{3/2}}{\theta^{7/2}} \right) \right\} \]

Now as \( T \to T_c^+ \) from above, \( \theta \to 0 \), we have
\( \theta^{3/2} \) and \( \theta^{5/2} \) are finite, but \( \theta^{7/2} \) and \( \theta^{9/2} \) \( \to \infty \)

\[
\Rightarrow \text{ at } T_c^+ \\
\frac{d}{dT} \left( \frac{C_V}{Nk_B} \right) = -\frac{45}{8T_c} \frac{\theta^{5/2}}{\theta^{3/2}} - \frac{27}{8T_c} \frac{\theta^{7/2}}{\theta^{9/2}} \]

Now from Pathria Appendix D Eq.(8)
\( \theta^{3/2} \) = \( \lim_{a \to 0} \frac{\Pi(1 - v)}{a^{1/2}} \)

So
\( \frac{\theta^{7/2}}{\theta^{9/2}} \) = \( \lim_{a \to 0} \frac{\Pi(3/2)}{a^{1/2}} \left( \frac{a^{1/2}}{\Pi(1/2)} \right)^3 = \frac{\Pi(3/2)}{\left[ \Pi(1/2) \right]^3} \)

\[
= \frac{1}{2} \frac{\Pi^{1/2}}{\Pi^{3/2}} = \frac{1}{2} \frac{\Pi}{2\Pi} \quad \text{since} \quad \Pi(1/2) = \sqrt{\pi} \quad \Pi(3/2) = \frac{1}{2} \sqrt{\pi} \]

\[
\frac{d}{dT} \left( \frac{C_V}{Nk_B} \right) = \frac{45}{8} \cdot \frac{1.341}{2.612} \cdot \frac{1}{T_c} - \frac{27}{8} \left( \frac{2.612}{2\Pi} \right)^2 \frac{1}{T_c} \]

\[
= \frac{2.89}{T_c} - \frac{3.66}{T_c} = -0.77 \]

\[
\frac{d}{dT} \left( \frac{C_V}{Nk_B} \right) = -\frac{0.77}{T_c}, \quad T = T_c^+ \]

The slope of \( C_V \) is discontinuous at \( T_c \).
CV has a cusp at T_c.

cV has a derivative & discontinuity at T_c goes to classical \( \frac{3}{2} \) as \( T \to \infty \)

\[
\frac{dCV}{dT} > 0 \text{ for } T = T_c^- \\
\frac{dCV}{dT} < 0 \text{ for } T = T_c^+
\]

**Entropy**

For a single species gas, we had for Gibbs free energy:

\[ G = N \mu \]

Also, \[ G = E - TS + pV \] (since G is Legendre transform of \( E \) with respect to \( S \) and \( V \)).

\[ \Rightarrow N \mu = E - TS + pV \]

\[ S = \frac{E + pV - N\mu}{T} \]

\[
\frac{S}{Nk_B} = \frac{E + pV}{Nk_BT} - \frac{\mu}{k_B T}
\]

We had earlier \( E = \frac{5}{2} pV \Rightarrow pV = \frac{2}{3} E \)

\[
\frac{S}{Nk_B} = \frac{5}{3} \frac{E}{N} \frac{1}{k_B T} - \frac{\mu}{k_B T}
\]
\[ Z = e^{\frac{E}{k_B T}} \]

\[ Z = 1 \text{ for } T < T_c \]

We had earlier:

\[ \frac{E}{N} = \frac{3}{2} \frac{k_B T}{\lambda^3} g_{3/2}(z) \]

and

\[ m = \frac{1}{\nu} = g_{3/2}(z) \frac{1}{\lambda^3} \text{ for } T > T_c \]

\[ \Rightarrow \frac{S}{N k_B} = \frac{5}{2} \frac{\nu}{\lambda^3} g_{3/2}(z) - \ln z = \begin{cases} \frac{5}{2} \frac{g_{5/2}(z)}{g_{3/2}(z)}, & T > T_c \\ \frac{5}{2} \frac{\nu}{\lambda^3} g_{3/2}(1), & T \leq T_c \end{cases} \]

Note: For \( T \leq T_c \) we had that the density \( g_{3/2}(z) \) in the normal state (i.e., the density of excited particles) \( \frac{g_{3/2}(1)}{\lambda^3} \leq m \).

\[ \Rightarrow \text{ for } T < T_c, \quad \frac{S}{N k_B} = \frac{5}{2} \left( \frac{m_n}{m} \right) \frac{g_{5/2}(1)}{g_{3/2}(1)} \rightarrow 0 \text{ as } T \rightarrow 0 \]

We can imagine that each normal particle carries

\[ \text{entropy } \frac{5}{2} k_B \frac{g_{5/2}(1)}{g_{3/2}(1)} \text{ per particle} \]

is just the above entropy per "normal" particle times the fraction of normal particles.

\[ \Rightarrow \text{ normal particle carries zero entropy} \]

\[ \text{entropy difference per particle between normal state and condensate state is } \Delta S = \frac{5}{2} k_B \frac{g_{5/2}(1)}{g_{3/2}(1)} \]
latent heat of condensation

\[ L = T \Delta S = \frac{5}{2} k_B T \frac{g_{\text{vap}}(1)}{g_{\text{vap}}(1)} \]

energy released upon converting one normal particle to one condensate particle.

⇒ mixed phase is like coexistence region of a 1st order phase transition (like water ↔ ice) - need to remove energy to turn water to ice.

⇒ "two fluid" model of mixed region.