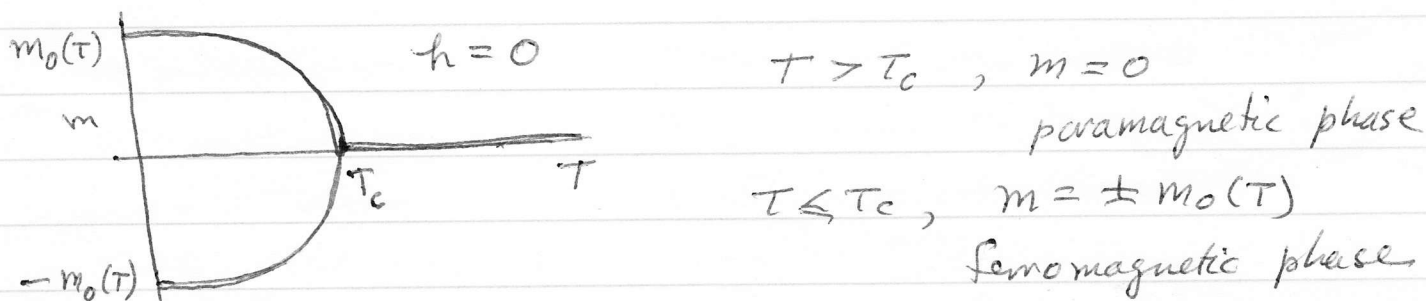


If such broken symmetry states exist at finite T , then do they persist at all T ? or do they disappear at a well defined T_c ?

Possibility of a phase transition

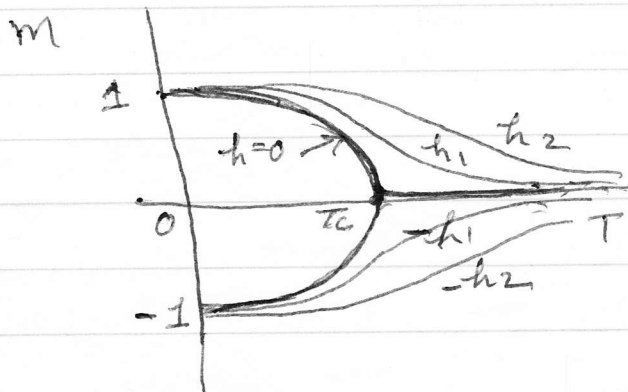


$m(T, 0)$ is singular at $T = T_c$

T_c is ferromagnetic phase transition

The ordered state at $T \leq T_c$ is a state of spontaneously broken symmetry. In $h = 0$ the system will pick either the up or the down state to order in, breaking the symmetry of the Hamiltonian.

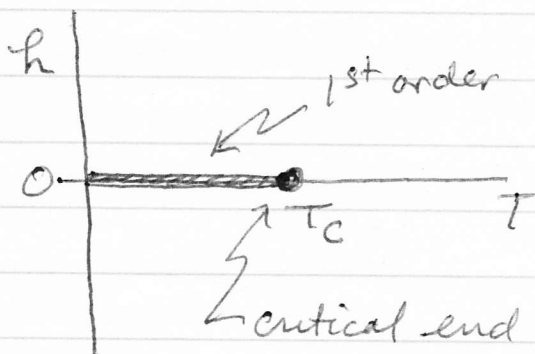
At finite h , expect $m(T, h)$ to behave like



$h_1 < h_2$

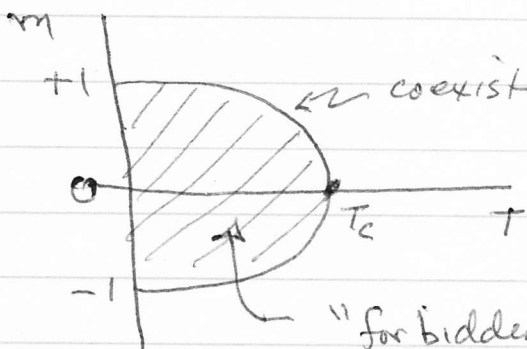
$m(T, h)$ is smooth function of T for $h \neq 0$.

Phase diagram in $h-T$ plane



1st order phase transition. As cross this line decreasing h , $m(T, h)$ has a discontinuous jump from $m_0(T)$ to $-m_0(T)$.
 coexistence line - phases $+m_0$ and $-m_0$ can coexist together
 critical end point. $m(T, h)$ is continuous if cross $h=0$ line above T_c . We will see that T_c corresponds to a 2nd order phase transition - jump in $m(T, h)$ vanishes continuously as approach T_c from below.

Phase diagram in $m-T$ plane



coexistence curve. "up" and "down" states at $h=0$ can coexist in equilib along this line
 "forbidden region" - there is no homogeneous phase with T and m in this region
 "phase separation region" - if cool a system with fixed M into this region it will phase separate into domains of "up" and "down" with average magnetization M .

Many similarities to liquid-gas phase diagram

We said that to have a state of spontaneously broken symmetry at finite T ~~requires~~, one needs to be in thermodynamic limit $N \rightarrow \infty$.

Similarly, true singular phase transitions can only occur in this $N \rightarrow \infty$ limit. Proof as follows:
partition function sum:

$$Z(T, h) = \sum_{\{s_i\}} e^{-\beta H[s_i]}$$

For finite system (N finite) the number of configurations to sum over is 2^N is finite.

Z is therefore the sum of a finite number of analytic functions ("analytic" here in the sense of complex function theory - has no singularities as vary T, h). As such, Z must itself be an analytic function of T and h .

$\Rightarrow Z$ can have no singularities

\Rightarrow no singularities in any thermodynamic quantities

\Rightarrow no phase transitions.

Only in thermodynamic limit of $N \rightarrow \infty$ is Z now the sum of an infinite ~~many~~ number of analytic functions. Such an infinite sum need NOT be analytic, so phase transitions can exist.

Having outlined what we might expect from the Ising model, we would now like to compute properties and see what happens!

However, an exact solution is not in general possible. Exact solutions to Ising model exist in

$d=1$ dimension - we will do this later

$d=2$ dimension - famous solution by Onsager

in $d=3$ dimensions, the best one has is very accurate numerical simulations - no exact solution.

⇒ Approximate Solution

Mean Field or Curie-Weiss Molecular Field Approximation

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} S_i S_j - h \sum_i S_i$$

Consider spin S_i . Approx the interaction of S_i with its neighbor S_j as an interaction with the thermal average value of $\langle S_j \rangle \equiv m$.

Instead of S_i seeing the specific S_j that vary from config to config, we say S_i sees only the effective average value of S_j - which is the same value m for all sites j . $\langle S_j \rangle = m = \frac{1}{N} \sum_i \langle S_i \rangle$

$$\mathcal{H}_{MF} \approx -J \sum_{\langle i,j \rangle} S_i m - h \sum_i S_i$$

↑
sum over bonds

$$H_{MF} = -\frac{z}{2} J \sum_i s_i m - h \sum_i s_i$$

$$= -\left(\frac{z}{2} J m + h\right) \sum_i s_i$$

each bond is shared
by two sites - gives
the factor $\frac{1}{2}$

where z is the "coordination number" - the number of
nearest neighbors of site i . For a simple (3d)
cubic lattice, $z=6$. For a ^(2d) square lattice, $z=4$.

In this approx, the interaction of s_i with its
neighbors is just like the interaction of s_i with
an additional average magnetic field $\frac{z}{2} J m$
- hence the origin of the name "mean field" approx.

To complete the approx, we need to self-consistently
compute m using H_{MF} and self-consistently solve
for m from the resulting equation.

$$H_{MF} = \sum_i H_{MF}^{(i)} \quad \text{where } H_{MF}^{(i)} = -\left(\frac{z}{2} J m + h\right) s_i$$

we have non-interacting spins in MF approx

$$\Rightarrow \langle s \rangle = m = \frac{\sum_s e^{-\beta H_{MF}^{(i)}} s}{\sum_s e^{-\beta H_{MF}^{(i)}}}$$

s is a single spin
at any site

$$= \frac{e^{+\beta(\frac{z}{2} J m + h)} + e^{-\beta(\frac{z}{2} J m + h)}}{e^{+\beta(\frac{z}{2} J m + h)} + e^{-\beta(\frac{z}{2} J m + h)}}$$

$$m = \tanh \left[\beta \left(\frac{z}{2} J m + h \right) \right]$$

solve to get
 $m(T, h)$

Note $m(T, h) = -m(T, -h)$ as expected

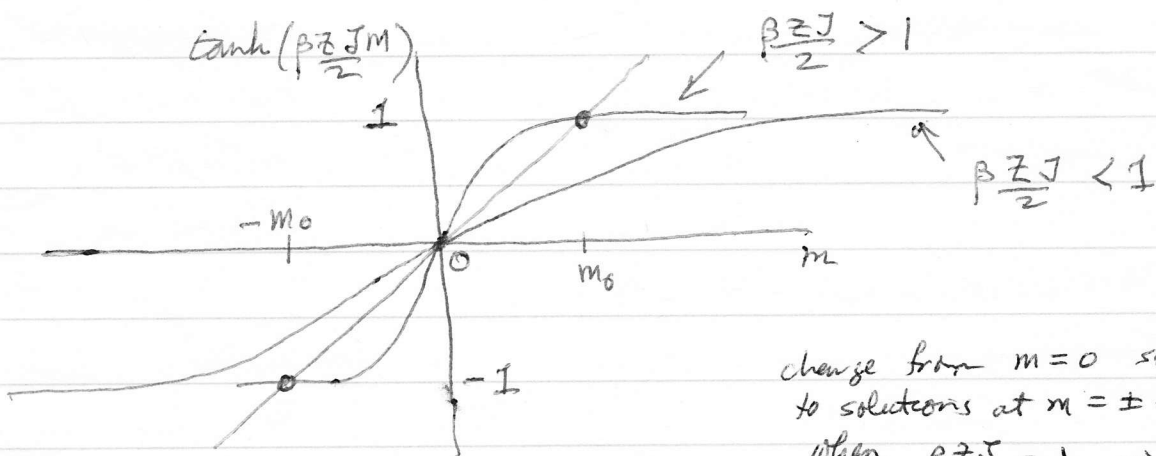
$$m = \tanh \left[\frac{\beta Z J m}{2} + \beta h \right]$$

consider first $h=0$

$$m = \tanh \left[\frac{\beta Z J m}{2} \right] \quad \text{could solve graphically}$$

since $\tanh x = x - \frac{1}{3}x^3 + O(x^5)$ we see that for $\frac{\beta Z J}{2} < 1$, the only solution will be $m=0$.

However for $\frac{\beta Z J}{2} > 1$ there are two additional solutions $m = \pm m_0$



change from $m=0$ solution only to solutions at $m = \pm m_0$ also when $\frac{\beta Z J}{2} = 1 \Rightarrow k_B T_c = \frac{Z J}{2}$

$$\Rightarrow \text{critical temperature } \boxed{k_B T_c = \frac{Z J}{2}}$$

$$T > T_c \Rightarrow m = 0$$

$$T < T_c \Rightarrow m = \pm m_0$$

$m=0$ is unstable solution