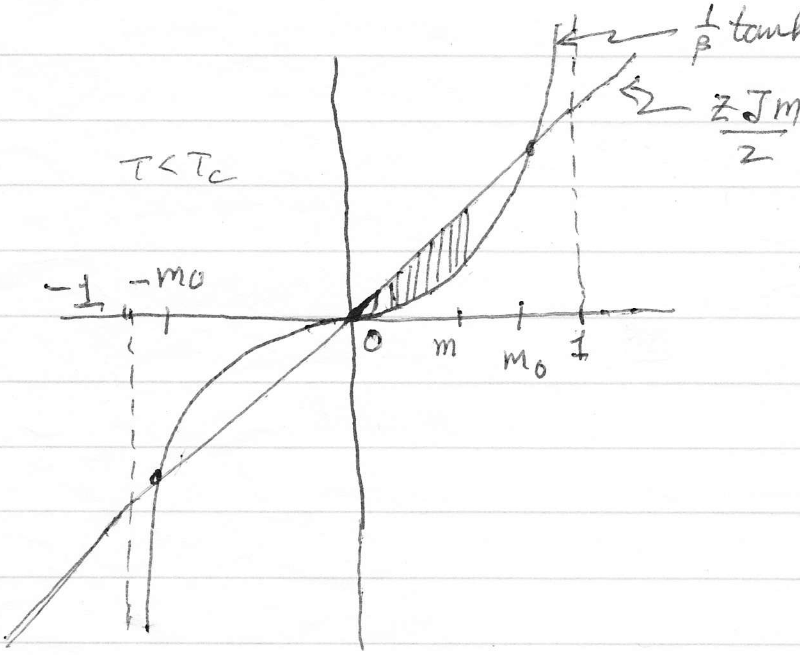
For $T < T_c$, $m = 0$ is unstable
$m = \pm m_0$ are the equilib solutions. To see this

$$m = \tanh\left(\frac{\beta z J}{2} m + \beta h\right)$$

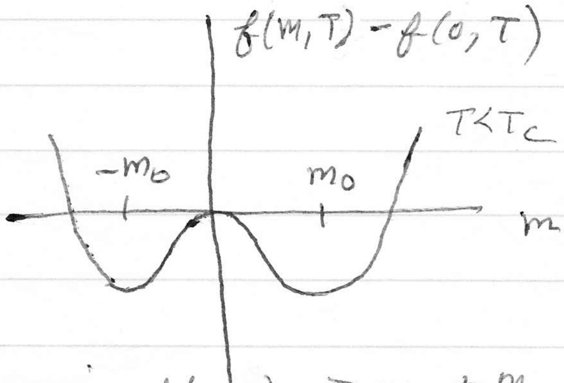
$$h = \frac{1}{\beta}\tanh^{-1} m - \frac{zJm}{2}$$

$$\left(\frac{\partial f}{\partial m}\right)_T = h \Rightarrow f(m,T) = \int_0^m h(m')\,dm' + f(0,T)$$



$\int_0^m h(m')\,dm'$ is the negative of the shaded area shown to the left. We see this area increases in magnitude until $m = m_0$, and then decreases in magnitude as m exceeds m_0 (since the curves cross at m_0)

Therefore we can plot the free energy $f(m,T) - f(0,T)$



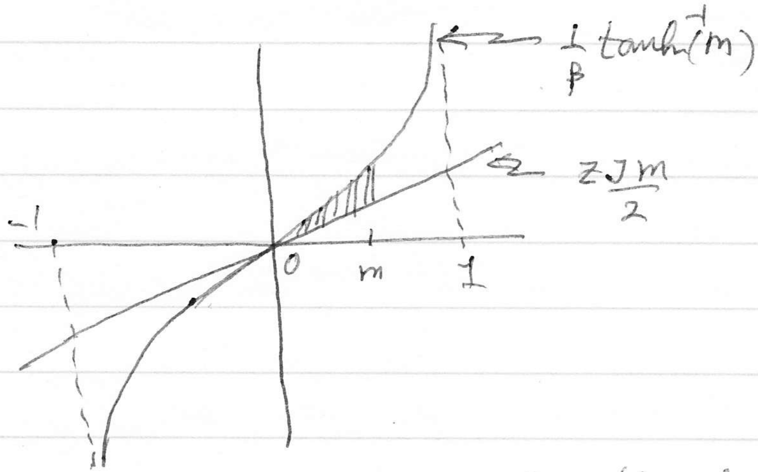
so $f(m_0,T) < f(0,T)$

m_0 gives the min of the free energy and so is the equilib solution

Gibbs free energy

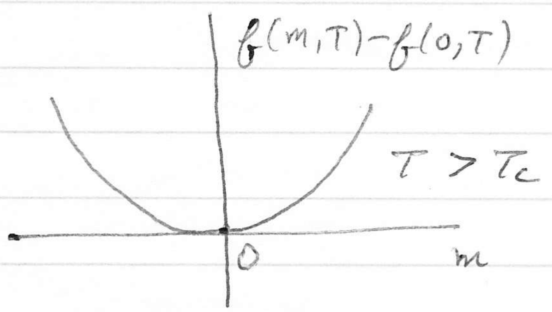
$$g(h=0,T) = \min_m f(m,T) \Rightarrow m = \pm m_0$$

For $T > T_c$ the situation looks like



now $\int_0^m h(m')\,dm'$ is the positive of the area shown to the left - it increases monotonically as m increases

so the free energy looks like



$\Rightarrow m=0$ is minim of $f(m,T)$

$$g(h=0,T) = \min_m f(m,T)$$

$\Rightarrow m=0$ is equilib state

We can examine these points analytically if we consider behavior near T_c where m is small.
This analysis will introduce the critical exponents δ, β, γ that characterize the critical point at $(T_c, h=0)$

$$m = \tanh\left(\beta \frac{zJ}{2} m + \beta h\right)$$

use $\frac{zJ}{2} = k_B T_c$, $\tanh x \simeq x - \frac{1}{3}x^3$ for small x

for small h, near T_c where m small, expand the tanh

$$m = \left(\frac{T_c}{T} m + \frac{h}{k_B T}\right) - \frac{1}{3}\left(\frac{T_c}{T} m + \frac{h}{k_B T}\right)^3$$

for small $\frac{h}{k_B T} \ll m$, expand the second term to $O(h)$

$$m = \left(\frac{T_c}{T} m + \frac{h}{k_B T}\right) - \frac{1}{3}\left(\frac{T_c}{T}\right)^3 m^3 - \left(\frac{T_c}{T}\right)^2 m^2 \frac{h}{k_B T}$$

(*) $$m\left(1 - \frac{T_c}{T}\right) + \frac{1}{3}\left(\frac{T_c}{T}\right)^3 m^3 = \frac{h}{k_B T}\left(1 - \left(\frac{T_c}{T}\right)^2 m^2\right)$$

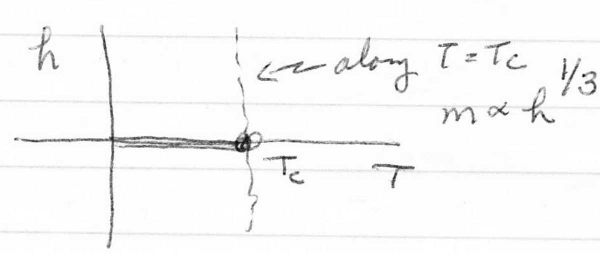
$$h = k_B T \left\{ \frac{m\left(1 - \frac{T_c}{T}\right) + \frac{1}{3}\left(\frac{T_c}{T}\right)^3 m^3}{1 - \left(\frac{T_c}{T}\right)^2 m^2} \right\}$$

(**) $$\boxed{h \simeq k_B T \left\{ m\left(1 - \frac{T_c}{T}\right) + \left[\left(1 - \frac{T_c}{T}\right)\left(\frac{T_c}{T}\right)^2 + \frac{1}{3}\left(\frac{T_c}{T}\right)^3\right] m^3 \right\}}$$

1) At $T = T_c$ critical isotherm

$$h = \frac{k_B T_c}{3} m^3 \propto m^\delta \qquad \delta = 3$$

or $m \propto h^{1/\delta}$



(2) At $h=0$ on coexistence line

from (*) with $h=0$ we have

$$\left(1-\frac{T_c}{T}\right)m + \left[\frac{1}{3}\left(\frac{T_c}{T}\right)^3 + \cancel{\left(\frac{T_c}{T}\right)\left(\frac{T_c}{T}\right)}\right]m^3 = 0$$

as $T \to T_c^-$, where $|m|>0$, $\left(1-\frac{T_c}{T}\right) + \frac{1}{3}m^2 = 0$

$$m = \pm\sqrt{\frac{3(T_c - T)}{T}}$$

Define $t = \frac{T_c - T}{T_c}$ $\quad m \propto \pm\sqrt{3t} \propto t^\beta \quad \beta = 1/2$

(3) At $h=0$ on coexistence line as $T \to T_c$

from (**)

$$\frac{\partial h}{\partial m} = k_B T\left\{\left(1-\frac{T_c}{T}\right) + 3\left[\left(1-\frac{T_c}{T}\right)\left(\frac{T_c}{T}\right)^2 + \frac{1}{3}\left(\frac{T_c}{T}\right)^3\right]m^2\right\}$$

$$\simeq k_B T\left\{\left(1-\frac{T_c}{T}\right) + m^2\right\} \quad \text{as } T \to T_c$$

As $T \to T_c^+$ from above, $m = 0$

$$\Rightarrow \frac{\partial h}{\partial m} = k_B T\left(1 - \frac{T_c}{T}\right) = k_B(T - T_c)$$

magnetic susceptibility $\Rightarrow$ $\frac{\partial m}{\partial h} = \chi^+ = \frac{1}{k_B(T-T_c)} \propto \frac{1}{|t|^\gamma}, \quad \gamma = 1$

Note: at high temp $T \gg T_c$, $\chi \sim \frac{1}{T}$ just like in Curie paramagnetism. Hence we say the $T > T_c$ phase is paramagnetic.

As $T \to T_c^-$ from below, $m^2 = 3\left(\frac{T_c - T}{T}\right)$

$$\Rightarrow \frac{\partial h}{\partial m} = k_B T\left(\left(1 - \frac{T_c}{T}\right) + 3\left(\frac{T_c - T}{T}\right)\right)$$

$$= 2k_B\,(T_c - T)$$

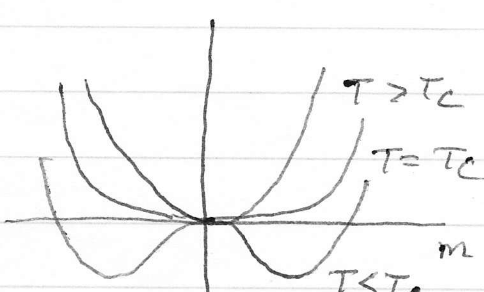
$$\frac{\partial m}{\partial h} = \chi^- = \frac{1}{2k_B(T_c - T)} \propto \frac{1}{|t|^\gamma} \qquad \gamma = 1$$

when h = 0

Also $\lim_{T \to T_c}\left(\frac{\chi^+}{\chi^-}\right) = \frac{2k_B(T_c - T)}{k_B(T - T_c)} = 2$ ← amplitude ratio

free energy $f(m,T) - f(0,T) = \int_0^m h(m')\,dm'$ use $(**)$ as $T \to T_c$

$$\Rightarrow f(m,T) - f(0,T) = k_B T\left\{\frac{1}{2}\left(1 - \frac{T_c}{T}\right)m^2 + \frac{1}{12}m^4\right\}$$



coefficient of m^2 term vanishes at T_c, goes negative below $T_c \Rightarrow$ minimum of $f(m,T)$ changes from $m = 0$ to $m = \pm m_0(T)$

$g(h=0,T) = \min_m f(m,T) \Rightarrow$ min of f gives equilibrium state when h = 0

4) Specific heat at $h=0$ along 1st order transition line

From above we can write

$$f(m,T) - f(0,T) = k_B T\left[\frac{1}{2}\left(1-\frac{T_c}{T}\right)m^2 + \frac{1}{12}m^4\right]$$

$$\equiv am^2 + bm^4$$

with $a = a_0(T-T_c)$ and $a_0 = \frac{k_B}{2}$

$b = \frac{k_B T}{12} \simeq \frac{k_B T_c}{12}$

Then for $T > T_c$ $m_0^2 = 0$

for $T < T_c$ $m_0^2 = -\frac{a}{2b}$ at minimum of $f(m,T) - f(0,T)$

since we want the specific heat at h=0, we need to work with the Gibbs free energy g(h,T), rather than the Helmholtz free energy f(m,T)

$$g(h,T) = \min_m\left[f(m,T) - mh\right]$$

so

$$g(h=0,T) = \min_m\left[f(m,T)\right] = f(m_0,T)$$

$T > T_c$ $g(h=0,T) = f(m_0,T) = f_0(T)$ as $m_0 = 0$

$T < T_c$ $g(h=0,T) = f(m_0,T) = f_0(T) + a m_0^2 + b m_0^4$

$$= f_0(T)\, a\left(-\frac{a}{2b}\right) + b\left(-\frac{a}{2b}\right)^2$$

$$= f_0(T) - \frac{a^2}{2b} + \frac{a^2}{4b}$$

$$= f_0(T) - \frac{a^2}{4b}$$

with $a = a_0(T-T_c)$

and $b = \frac{k_B T_c}{12}$ constant

specific heat at $h=0$

specific heat per spin

$$s = -\left(\frac{\partial g}{\partial T}\right)_{h=0} \Rightarrow C \equiv T\left(\frac{\partial s}{\partial T}\right)_{h=0} = -T\left(\frac{\partial^2 g}{\partial T^2}\right)_{h=0}$$

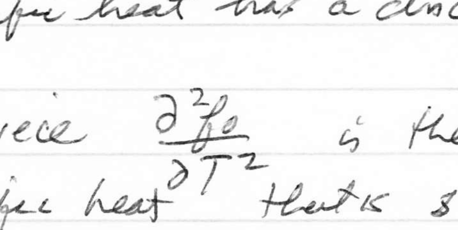
$$C = -T\,\frac{d^2 f(m_0(T),T)}{dT^2}$$

$$= -T\,\frac{d^2 f_0}{dT^2} \qquad T > T_c \quad \text{where } m_0 = 0$$

$$= -T\,\frac{d^2 f_0}{dT^2} + \frac{T a_0^2}{2b} \qquad T < T_c \quad \text{where } m_0^2 = \frac{-a}{2b}$$

since $\frac{d a^2}{dT^2} = 2a_0^2$

$$\Delta C = C(T \to T_c^-) - C(T \to T_c^+) = \frac{T_c a_0^2}{2b}$$



Specific heat has a discontinuous jump at T_c

The piece $\frac{\partial^2 f_0}{\partial T^2}$ is the non-singular piece of the specific heat that is smooth and continuous as one passes through T_c.

We can define a critical exponent α for the specific heat by

$$C \propto |t|^{-\alpha} \qquad \text{or}$$

$$\alpha = \lim_{t \to 0} \left[\frac{\ln C}{\ln|t|} \right]$$

For our mean field calculation this gives $\boxed{\alpha = 0}$

Summary: Critical exponents for Ising model in mean-field theory

$T<T_c,\ h=0 \qquad m_0(T) \sim |t|^{\beta} \qquad \beta = 1/2$

$T = T_c \qquad h(m) \sim m^{\delta} \qquad \delta = 3$

$h=0 \qquad \chi(T) \sim \frac{1}{|t|^{\gamma}} \qquad \gamma = 1$

$\lim_{t\to 0}\left(\frac{\chi^+}{\chi^-}\right) = 2$ amplitude ratio

$h=0 \qquad C(T) \sim |t|^{-\alpha} \qquad \alpha = 0$

exponent values in mean field theory are independent of the dimension d of the system.

From exact Onsager solution of $d=2$ Ising model

$\beta = 1/8$, $\delta = 15$, $\gamma = 7/4$, $\alpha = 0$ but C has logarithmic divergence $C \sim \ln|t|$