Unit 3-7: Black Body Radiation

Cavity radiation:

A volume V at fixed temperature T absorbs and emits electromagnetic radiation. What are the characteristics of this equilibrium radiation at fixed T?

Electromagnetic waves with wavevector $\mathbf{k}$ have frequency $\omega = c|\mathbf{k}|$, with two transverse polarizations for each $\mathbf{k}$. (there is no longitudinal polarization for EM waves).

Regard each mode of electromagnetic wave as an oscillator. If excited to energy level n, the energy in the oscillator of wavevector $\mathbf{k}$ is $\epsilon_{\mathbf{k}} = n\hbar\omega = n\hbar c|\mathbf{k}| \;\Rightarrow\;$ *n photons* in this mode. The average energy in this mode is therefore,

$$\langle\epsilon_{\mathbf{k}}\rangle = \hbar\omega\langle n\rangle = \frac{\hbar\omega}{e^{\beta\hbar\omega}-1} \tag{3.7.1}$$

(we ignore the ground state energy $\frac{1}{2}\hbar\omega$ as it is a temperature independent constant.)

For a volume $V = L^3$, periodic boundary conditions give the allowed wavevectors as $\mathbf{k} = \left(\frac{2\pi}{L}\right)\mathbf{n}$, with $\mathbf{n} = (n_x, n_y, n_z)$ integer.

As we did for phonons, we can now compute the density of states $g(\omega)$, *per unit volume*, for photons with frequency less than or equal to ω. The calculation is exactly the same as we did for phonons in Notes 3-6, except that (i) we replace the speed of sound c_s by the speed of light in the vacuum c, and (ii) for photons there are only two transverse polarization for each $\mathbf{k}$, whereas for phonons there were three polarizations (two transverse and one longitudinal). So the density of states for photons is just 2/3 the density of states for phonons, with $c_s \to c$. From Eq. (3.6.11) we therefore have,

$$g(\omega) = \frac{1}{\pi^2}\frac{\omega^2}{c^3} \tag{3.7.2}$$

Classically, each electromagnetic mode of oscillation would be like a classical harmonic oscillator, and so by the equipartition theorem it would contribute k_BT to the average energy. The classical prediction for the average energy per volume at frequency ω would then be,

$$u^{\text{class}}(\omega) = g(\omega)k_BT = \frac{1}{\pi^2}\frac{\omega^2}{c^3}k_BT \tag{3.7.3}$$

or in terms of the wavelength $\lambda = 2\pi c/\omega$,

$$u^{\text{class}}(\lambda) = u^{\text{class}}(\omega)\left|\frac{d\omega}{d\lambda}\right| = \frac{8\pi}{\lambda^4}k_BT \tag{3.7.4}$$

Thus the amount of energy in the high frequency $\omega \to \infty$, or in the low wavelengths $\lambda \to 0$, grows without bound. This was contrary to experimental observation. Moreover, since (unlike for phonons in a solid) there is no upper bound on the possible frequency ω (or lower bound on the wavelength λ), so when one computes the total energy in all modes $\int_0^\infty d\omega\, u^{\text{class}}(\omega) = \int_0^\infty d\lambda\, u^{\text{class}}(\lambda)$, this will diverge, and the specific heat will also diverge. This was known as the *ultraviolet catastrophe*, because the divergence comes from the behavior at large ω or equivalently at small λ.

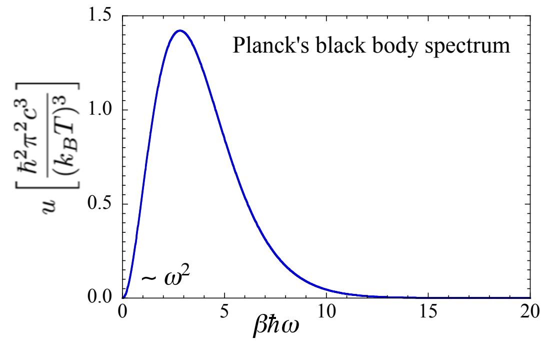
The resolution of this paradox came by understanding that we must quantize the oscillations of the electromagnetic waves. In this case, The average energy per volume $u(\omega)$ at frequency ω is,

$$u(\omega) = g(\omega)\hbar\omega\langle n(\omega)\rangle = g(\omega)\left(\frac{\hbar\omega}{e^{\beta\hbar\omega}-1}\right) \tag{3.7.5}$$

This follows since $\hbar\omega$ is the energy of a single photon of frequency ω, $\langle n(\omega)\rangle$ is the average number of photons in such a mode at ω, and $g(\omega)$ is the number of modes per unit energy per unit volume at ω.

Substituting in for $g(\omega)$ then gives,

$$\boxed{u(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3\left(\mathrm{e}^{\beta\hbar\omega}-1\right)}} \tag{3.7.6}$$



This is Planck's formula for the Black Body Spectrum. Fitting experimental data to this form is how Planck first measured the constant $h = 2\pi\hbar$!

Note, for low frequencies such that $\beta\hbar\omega \ll 1 \;\Rightarrow\; \hbar\omega \ll k_BT$, the Planck formula of Eq. (3.7.6) reduces to the classical result in Eq. (3.7.3). But for high frequencies, such that $\hbar\omega > k_BT$, the Planck distribution reaches a peak and then decreases exponentially as ω increases. This is what avoids the ultraviolet catastrophe. It is Planck's constant $\hbar$ that determines the crossover from classical behavior at low $\omega \ll k_BT/\hbar$, to quantum behavior at high $\omega \gg k_BT/\hbar$.

Total energy density:

The total energy density is then,

$$\frac{E}{V} = \int_0^\infty d\omega\, u(\omega) = \frac{\hbar}{\pi^2c^3}\int_0^\infty d\omega\,\frac{\omega^3}{\mathrm{e}^{\beta\hbar\omega}-1} = \frac{\hbar}{\pi^2c^3}\frac{1}{(\beta\hbar)^4}\int_0^\infty dx\,\frac{x^3}{e^x-1} \qquad \text{with } x=\beta\hbar\omega \tag{3.7.7}$$

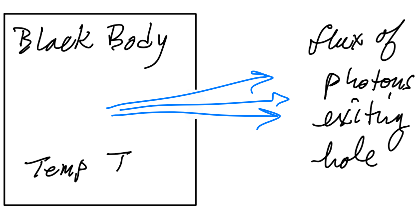
The integral over x just give the constant $\pi^4/15$, so we have,

$$\frac{E}{V} = \left(\frac{\pi^2 k_B^4}{15\,\hbar^3c^3}\right)T^4 \tag{3.7.8}$$

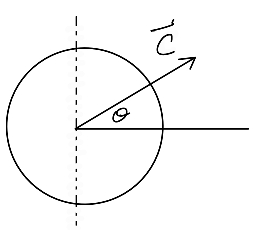
Note: A big difference between photons and phonons is that for phonons there is a largest possible $|\mathbf{k}| = k_D$ set by the spacing between the ions in the lattice. But for photons there is no such maximum $|\mathbf{k}|$.

Energy flux from a cavity:

Now consider the flux of energy exiting from a hole in a cavity. We have for the flux $\mathcal{F}$,



$$\begin{aligned}\mathcal{F} &= \text{(energy density)(speed)(projection of velocity out the hole)}\\ &= \left(\frac{E}{V}\right)c\langle\cos\theta\rangle\end{aligned} \tag{3.7.9}$$



We have,

$$\langle\cos\theta\rangle = \frac{1}{4\pi}\int_0^{2\pi}d\varphi\int_0^{\pi/2}d\theta\,\sin\theta\,\cos\theta = \frac{2\pi}{4\pi}\left(\frac{\sin^2\theta}{2}\right)_0^{\pi/2} = \frac{1}{4} \tag{3.7.10}$$

Note, the integral on θ goes only to $\pi/2$ since, when $\theta > \pi/2$, the particle is traveling *away* from the hole.

So,

$$\mathcal{F} = \left(\frac{E}{V}\right)\frac{c}{4} = \frac{\pi^2k_B^4}{60\,\hbar^3c^2}T^4 = \sigma T^4 \qquad \text{Stefan-Boltzmann Law} \tag{3.7.11}$$

$\sigma = \dfrac{\pi^2k_B^4}{60\,\hbar^3c^2} = 5.7\times10^{-8}\,W/m^2K^4$ is Stefan's constant.

Pressure of a photon gas:

We have,

$$\frac{p}{k_BT} = \frac{1}{V}\ln\mathcal{L} = -\frac{1}{V}\sum_s\sum_{\mathbf{k}}\ln\left(1-\mathrm{e}^{-\beta\epsilon_{\mathbf{k}}}\right) \qquad \text{BE partition function with } \mu = 0 \tag{3.7.12}$$

$$= -\frac{2}{V}\sum_{\mathbf{k}}\ln\left(1-\mathrm{e}^{-\beta\epsilon_{\mathbf{k}}}\right) = -\int_0^\infty d\omega\, g(\omega)\ln\left(1-\mathrm{e}^{-\beta\hbar\omega}\right) \tag{3.7.13}$$

$$= -\frac{1}{\pi^2c^3}\int_0^\infty d\omega\,\omega^2\ln\left(1-\mathrm{e}^{-\beta\hbar\omega}\right) \tag{3.7.14}$$

We integrate by parts,

$$\frac{p}{k_BT} = -\frac{1}{\pi^2c^3}\left[\frac{\omega^3}{3}\ln\left(1-\mathrm{e}^{-\beta\hbar\omega}\right)\right]_0^\infty + \frac{1}{\pi^2c^3}\int_0^\infty d\omega\,\frac{\omega^3}{3}\,\frac{\beta\hbar\mathrm{e}^{-\beta\hbar\omega}}{1-\mathrm{e}^{-\beta\hbar\omega}} \tag{3.7.15}$$

The boundary term vanishes at both its limits: (i) as $\omega\to\infty$, $\ln(1-\mathrm{e}^{-\beta\hbar\omega})\to-\mathrm{e}^{-\beta\hbar\omega}$, so $\omega^3\ln(1-\mathrm{e}^{-\beta\hbar\omega})\to -\omega^3\mathrm{e}^{-\beta\hbar\omega}\to 0$, and (ii) as $\omega\to 0$, $\ln(1-\mathrm{e}^{-\beta\hbar\omega})\to\ln(\beta\hbar\omega)$ and so $\omega^3\ln(\beta\hbar\omega)\to 0$. We are left with,

$$\frac{p}{k_BT} = \frac{\beta\hbar}{3\pi^2c^3}\int_0^\infty d\omega\left(\frac{\omega^3}{\mathrm{e}^{\beta\hbar\omega}-1}\right) \tag{3.7.16}$$

Comparing to the calculation of E/V in Eq. (3.7.7) we have,

$$\frac{p}{k_BT} = \frac{\beta}{3}\frac{E}{V} = \frac{1}{3k_BT}\frac{E}{V} \qquad \Rightarrow \qquad \boxed{p = \frac{1}{3}\frac{E}{V}} \quad \text{pressure of a photon gas} \tag{3.7.17}$$

We can compare this to a non-relativistic ideal gas of identical particles, which has,

$$pV = Nk_BT, \quad E = \frac{3}{2}Nk_BT, \quad \Rightarrow \quad p = \frac{2}{3}\frac{E}{V} \qquad \text{for non-relativistic particles} \tag{3.7.18}$$

The difference is because the photons are relativistic particles, and as you showed in Discussion Question 3, the energy of such particles is related to temperature by $E = 3Nk_BT$. The ideal gas law still holds, and so one gets $\frac{1}{3}E = pV$.

The last two examples of phonons in a solid and black body radiation were problems involving bosons with a linear excitation spectrum, $\epsilon_{\mathbf{k}} = \hbar\omega_{\mathbf{k}} = \hbar c|\mathbf{k}|$, and zero chemical potential, $\mu = 0$.

Next we want to consider the problem of an ideal gas of non-interacting physical particles, bosons *or* fermions, with an ordinary quadratic non-relativisitic excitation spectrum, $\epsilon_{\mathbf{k}} = \hbar^2k^2/2m$, and with a finite chemical potential, $\mu \neq 0$.