Unit 3-2: Quantum Many Particle Systems – Bosons vs Fermions

A system of N identical (i.e., indistinguishable) particles is described by a wavefunction,

$$\psi(\mathbf{r}_1, s_1, \mathbf{r}_2, s_2, \ldots, \mathbf{r}_N, s_N) \equiv \psi(1, 2, \ldots, N) \qquad \text{where } \mathbf{r}_i \text{ and } s_i \text{ are the position and spin of particle } i \tag{3.2.1}$$

Identical particles means that the probability density $|\psi|^2$ should be symmetric under the interchange of any pair of coordinates,

$$|\psi(1, \ldots, i, \ldots, j \ldots, N)|^2 = |\psi(1, \ldots, j \ldots, i, \ldots, N)|^2 \tag{3.2.2}$$

There are two possible symmetries for ψ.

1) ψ is symmetric under pair interchanges, $\psi(1, \ldots, i, \ldots, j, \ldots, N) = \psi(1, \ldots, j \ldots, i, \ldots, N)$
2) ψ is antisymmetric under pair interchanges, $\psi(1, \ldots, i, \ldots, j, \ldots, N) = -\psi(1, \ldots, j \ldots, i, \ldots, N)$

Case (1) is called Bose-Einstein (BE) statistics. Particle that obey such statistics are called *bosons*.
Case (2) is called Fermi-Dirac (FD) statistics. Particles that obey such statistics are called *fermions*.

For a general permutation $\mathbb{P}$ that interchanges any number of pairs of particles,
For BE statistics, $\mathbb{P}\psi = \psi$.
For FD statistics, $\mathbb{P}\psi = (-1)^P\psi$, where P is the number of pairwise interchanges needed to make the permutation $\mathbb{P}$.
For FD, when P is even, then $\mathbb{P}\psi = +\psi$. When P is odd, then $\mathbb{P}\psi = -\psi$.

BE statistics are for particles with integer spin, $s = 0, 1, 2, \ldots$.
FD statistics are for particles with half integer spin, $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$.

Now consider *non-interacting* particles. The N-particle Hamiltonian is the sum of single particle Hamiltonians,

$$\mathcal{H}(1, 2, 3, \ldots, N) = \mathcal{H}^{(1)}(1) + \mathcal{H}^{(1)}(2) + \mathcal{H}^{(1)}(3) + \cdots + \mathcal{H}^{(1)}(N) \tag{3.2.3}$$

and we can write the N-particle wavefunction as a product of single particle wavefunctions,

$$\psi(1, 2, \ldots, N) = \phi_{i_1}(1)\phi_{i_2}(2) \cdots \phi_{i_N}(N) \tag{3.2.4}$$

where ϕ_i is an eigenstate of the single particle $\mathcal{H}^{(1)}$ with energy ϵ_i.

But while the above ψ will solve Schrodinger's equation, $\mathcal{H}\psi = E\psi$, with $E = \epsilon_{i_1} + \epsilon_{i_2} + \cdots + \epsilon_{i_N}$, this ψ does not have the proper symmetry required for BE or FD statistics. We can construct an appropriately symmetrized wavefunction as follows.

For BE,

$$\psi_{BE} = \frac{1}{\sqrt{N_P}} \sum_{\mathbb{P}} \mathbb{P}\,\psi \tag{3.2.5}$$

For FD,

$$\psi_{FD} = \frac{1}{\sqrt{N_P}} \sum_{\mathbb{P}} (-1)^P \mathbb{P}\,\psi \tag{3.2.6}$$

where the sum is over all permutations $\mathbb{P}$ of the N particles, N_P is the number of possible permutations of the N particles ($N_P = N!$), and ψ is the product of single particle wavefunctions as in Eq. (3.2.4).

You can verify that, with the above definitions, $\mathbb{P}\psi_{BE} = \psi_{BE}$, and $\mathbb{P}\psi_{FD} = (-1)^P\psi_{FD}$, for any permutation $\mathbb{P}$, as desired.

For a ψ described by Eq. (3.2.4), or its symmetrized versions ψ_{BE} and ψ_{FD}, the total energy is,

$$E = \epsilon_{i_1} + \epsilon_{i_2} + \cdots + \epsilon_{i_N} = \sum_j n_j \epsilon_j \tag{3.2.7}$$

where the last sum is over all single particle eigenstates ϕ_j, n_j is the number of particles in single particle eigenstate ϕ_j, and $\sum_j n_j = N$.

For BE statistics, $n_j = 0, 1, 2, \ldots$ is any integer.

For FD statistics, the only allowed possibilities are $n_j = 0$ or 1.

This is because if we had two particles in any given single particle state, say ϕ_1, then the wavefunction ψ would look like,

$$\psi(1, 2, 3, \ldots, N) = \phi_1(1)\phi_1(2)\phi_{i_3}(3)\cdots\phi_{i_N}(N) \tag{3.2.8}$$

But then when we construct $\psi_{FD} = \dfrac{1}{\sqrt{N_P}}\sum_{\mathbb{P}}(-1)^P\mathbb{P}\,\psi$, then for every term in the sum $\phi_1(i)\phi_1(j)\phi_{i_3}(k)\cdots\phi_{i_N}(\ell)$ there must also be a term $(-1)\phi_1(j)\phi_1(i)\phi_{i_3}(k)\cdots\phi_{i_N}(\ell)$ from interchanging $i \leftrightarrow j$, so these will cancel pair by pair and we find that $\psi_{FD} = 0$.

The Pauli Exclusion Principle: No two fermions can occupy the same single particle state; alternatively one could say, no two fermions can have the same "quantum numbers."

There is no similar restriction for bosons.

Occupation numbers: The specification of any *non-interacting* N particle quantum state can be given by the *occupation numbers* $\{n_i\}$, that give how many particles are in each single particle eigenstate ϕ_i. Each set of $\{n_i\}$ corresponds to *one* N-particle state.