Unit 4-4: Critical Exponents within the Mean-Field Approximation for the Ising Model

We can make the graphical presentation of the last section more analytical if we restrict our consideration to behavior near the critical temperature T_c, where m is always small. This analysis near T_c will then introduce the *critical exponents* that describe the nature of the singularity of the system at T_c, and we will find the values of these critical exponents within the approximate mean-field solution.

The self-consistent mean-field equation for the magnetization as a function of $\beta = 1/k_BT$ and applied magnetic field h is,

$$m = \tanh\left(\frac{\beta zJm}{2} + \beta h\right) \qquad \text{with} \qquad k_BT_c = \frac{zJ}{2} \tag{4.4.1}$$

For small h, near T_c where m is also small, we can expand the hyperbolic tangent, $\tanh x \approx x - \frac{1}{3}x^3$, to get,

$$m \approx \left(\frac{T_c}{T}m + \frac{h}{k_BT}\right) - \frac{1}{3}\left(\frac{T_c}{T}m + \frac{h}{k_BT}\right)^3 \tag{4.4.2}$$

For small $h/k_BT \ll m$, we can further expand the second term to $O(h)$ to get,

$$m = \left(\frac{T_c}{T}m + \frac{h}{k_BT}\right) - \frac{1}{3}\left(\frac{T_c}{T}\right)^3 m^3 - \left(\frac{T_c}{T}\right)^2 m^2 \frac{h}{k_BT} \tag{4.4.3}$$

which we can rearrange to write as,

$$\left(1 - \frac{T_c}{T}\right)m + \frac{1}{3}\left(\frac{T_c}{T}\right)^3 m^3 = \frac{h}{k_BT}\left[1 - \left(\frac{T_c}{T}\right)^2 m^2\right] \tag{4.4.4}$$

and then solve for h to get,

$$h = k_BT\left[\frac{\left(1 - \frac{T_c}{T}\right)m + \frac{1}{3}\left(\frac{T_c}{T}\right)^3 m^3}{1 - \left(\frac{T_c}{T}\right)^2 m^2}\right] \tag{4.4.5}$$

Since m is small, we can expand the denominator $1/(1-\delta) \approx 1 + \delta$, then keeping only terms up to $O(m^3)$ we get,

$$h = k_BT\left[\left(1 - \frac{T_c}{T}\right)m + \left[\left(1 - \frac{T_c}{T}\right)\left(\frac{T_c}{T}\right)^2 + \frac{1}{3}\left(\frac{T_c}{T}\right)^3\right] m^3\right] \tag{4.4.6}$$

Since we are only interested in behavior near T_c, we can now take $T \to T_c$ and keep only the leading order term in the coefficients of m and m^3. The above becomes,

$$h = k_BT\left[\left(1 - \frac{T_c}{T}\right)m + \frac{1}{3}m^3\right] \tag{4.4.7}$$

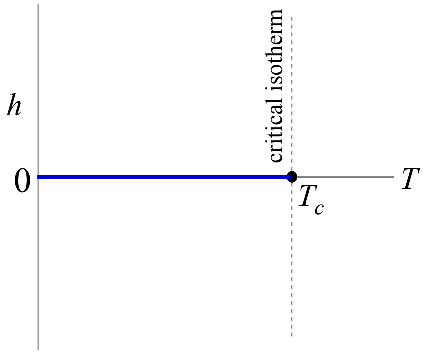
We can now define the critical exponents:

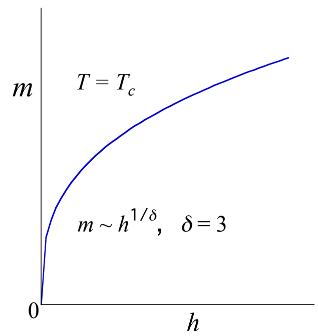
1) Magnetization along the critical isotherm $T = T_c$ and the critical exponent δ

The critical isotherm is the curve in the phase diagram at fixed $T = T_c$. Evaluating Eq. (4.4.7) at $T = T_c$ we get,

$$h = \frac{k_BT}{3}m^3 \propto m^\delta \qquad \text{or} \qquad m \propto h^{1/\delta} \qquad \text{at } T = T_c \qquad \text{with} \qquad \delta = 3 \tag{4.4.8}$$

At $T = T_c$, $m = 0$ when $h = 0$. If you then turn on a small magnetic field, the magnetization will grow as $m \propto h^{1/\delta}$. Since $1/\delta < 1$, the magnetic susceptibility $\chi = \lim_{h\to 0}\left(\frac{\partial m}{\partial h}\right)_T$ is therefore infinite at T_c.





2) <u>Magnetization along the coexistence curve at $h = 0$ and the critical exponent β</u>

The curve in the phase diagram for $T \leq T_c$ at $h = 0$ is the coexistence curve; in the $h - T$ plane this is just the line at $h = 0$ from $T = 0$ to $T = T_c$. Evaluating Eq. (4.4.7) for $T \leq T_c$ at $h = 0$ we get,

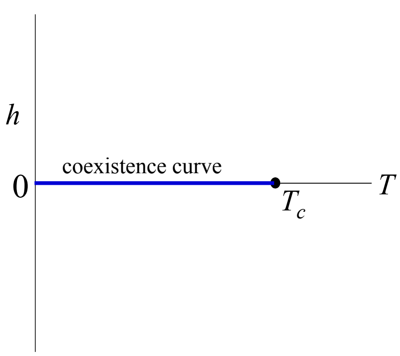
$$\left(1 - \frac{T_c}{T}\right) m + \frac{1}{3}m^3 = 0 \quad \Rightarrow \quad m = 0, \text{ or } \left(\frac{T_c - T}{T}\right) = \frac{1}{3}m^2 \quad \Rightarrow \quad m = 0, \text{ or } m = \pm\sqrt{\frac{3(T_c - T)}{T}} \tag{4.4.9}$$

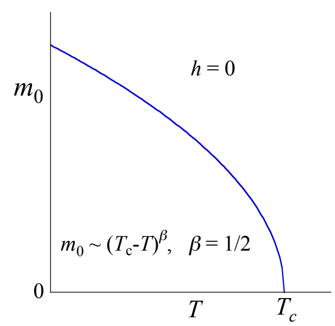
For $T > T_c$, the only real valued solution for the magnetization is $m = 0$. But for $T < T_c$, as we argued in the previous Notes 4-3, the equilibrium magnetization will be non-zero, and so given by the square root solutions.

Defining the *reduced temperature* $t \equiv (T_c - T)/T_c$, we have as $T \to T_c$ from below,

$$m = \pm\sqrt{3t} \propto t^\beta \qquad \text{with} \qquad \beta = 1/2 \tag{4.4.10}$$

Since we are at $h = 0$, the m here is just the spontaneous magnetization $\pm m_0(T)$ that the system develops in the ferromagnetic phase. We see, as mentioned earlier, that m_0 vanishes continuously as $T \to T_c$, and it vanishes with the power-law behavior $m_0 \propto t^\beta$.





3) <u>Magnetic susceptibility at $h = 0$ and the critical exponent γ</u>

We next consider the magnetic susceptibility $\chi = \lim\limits_{h\to 0}\left(\dfrac{\partial m}{\partial h}\right)_T$. From Eq. (4.4.7) we have,

$$\left(\frac{\partial h}{\partial m}\right)_T = k_B T\left[\left(1 - \frac{T_c}{T}\right) + m^2\right] \tag{4.4.11}$$

As $T \to T_c^+$ from above, we have $m = 0$, and so $\left(\dfrac{\partial h}{\partial m}\right)_T = k_B T\left(1 - \dfrac{T_c}{T}\right) = k_B(T - T_c)$. We therefore have,

$$\chi^+ = \lim_{h\to 0}\left(\frac{\partial m}{\partial h}\right)_T = \frac{1}{k_B(T - T_c)} \propto \frac{1}{|t|^{\gamma'}} \qquad \text{with} \qquad \gamma' = 1 \qquad \text{as } T \to T_c^+ \text{ from above} \tag{4.4.12}$$

Note, at large $T \gg T_c$, the above becomes $\chi^+ \propto \dfrac{1}{T}$ just like in Curie paramagnetism.

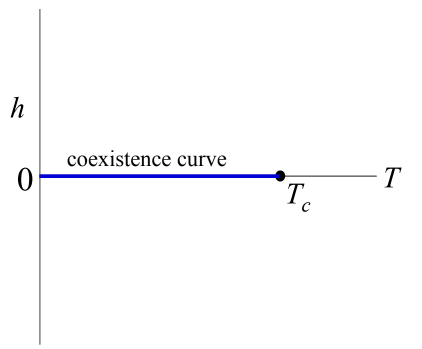
As $T \to T_c^-$ from below, we have $m^2 = 3\left(\dfrac{T_c - T}{T}\right)$, and so $\left(\dfrac{\partial h}{\partial m}\right)_T = k_B T\left[\left(1 - \dfrac{T_c}{T}\right) + 3\left(\dfrac{T_c}{T} - 1\right)\right] = 2k_B(T_c - T)$. We therefore have,

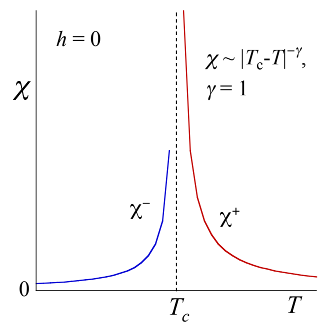
$$\chi^- = \lim_{h\to 0}\left(\frac{\partial m}{\partial h}\right)_T = \frac{1}{2k_B(T_c - T)} \propto \frac{1}{|t|^\gamma} \qquad \text{with} \qquad \gamma = 1 \qquad \text{as } T \to T_c^+ \text{ from below} \tag{4.4.13}$$

So we have $\gamma = \gamma'$; the magnetic susceptibility diverges with the same exponent whether we approach T_c from above or from below.

We also have,

$$\lim_{T\to T_c}\left|\frac{\chi^+}{\chi^-}\right| = \frac{2k_B|T_c - T|}{k_B|T - T_c|} = 2 \qquad \text{this is known as the amplitude ratio} \tag{4.4.14}$$





4) Free energy density and Landau Theory

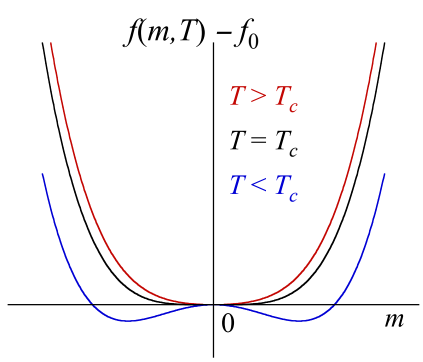
We can also integrate Eq. (4.4.7) to get the Helmholtz free energy density,

$$f(m,T) - f(0,T) = \int_0^m dm'\, h(m') = k_B T\left[\frac{1}{2}\left(1 - \frac{T_c}{T}\right)m^2 + \frac{1}{12}m^4\right] \tag{4.4.15}$$

We can write this in the form

$$f(m,T) = f_0 + am^2 + bm^4 \qquad \text{where, as } T \to T_c\text{, we have } a = a_0(T - T_c) \text{ and } b = \text{constant} \tag{4.4.16}$$

with $a_0, b > 0$, and $f_0 = f(0,T)$. The coefficient a is therefore positive for $T > T_c$, negative for $T < T_c$, and vanishes continuously as $T \to T_c$.



For $T > T_c$, with $a > 0$, we see that $f(m,T)$ has only a single minimum at $m = 0$. However, for $T < T_c$, with $a < 0$, the free energy has double minima at $m = \pm m_0 = \pm\sqrt{-a/2b}$. The is the spontaneous magnetization of the ferromagnetic state.

The Gibbs free energy density at $h = 0$ is given by $g(h = 0, T) = \min_m f(m,T)$. The equilibrium state has magnetization given by the minimizing value of m, i.e. $m = 0$ when $T > T_c$, and $m = \pm m_0$ when $T < T_c$.

The form of $f(m,T)$ in Eq. (4.4.16) is the starting point for the *Landau theory* of continuous phase transitions (Lev Landau). The Landau theory describes such transitions in terms of an *order parameter* m and an *ordering field* h. Turning on a finite h induces a finite m. When $h = 0$, one has $m = 0$

in the *disordered* phase, while $|m| > 0$ in the *ordered* phase with a spontaneous broken symmetry (hence the name order parameter). Near the ordering phase transition where m is small, Landau posited that one could expand the free energy $f(m,T)$ as a power series in m that is consistent with all the symmetries of the problem, keeping only the lowest order terms needed for a non-trivial behavior. For the Ising model at $h = 0$, the Hamiltonian has an inversion symmetry with respect to $s_i \leftrightarrow -s_i$, so the free energy must satisfy $f(m,T) = f(-m,T)$, and so only even powers appear in the expansion. If the coefficient of the quartic term is positive, and the coefficient of the quadratic term changes sign, that is then the minimal model to give a phase transition, as demonstrated above. For a vector spin model, such as the XY or Heisenberg model, the free energy would be $f(\mathbf{m},T) = f_0 + a|\mathbf{m}|^2 + b|\mathbf{m}|^4$. For other physical systems, $f(m,T)$ can have other forms – for example, for a system with a scalar order parameter but no inversion symmetry, we could have $f(m,T) = f_0 + f_1 m + f_2 m^2 + f_3 m^3 + f^4 m^4$ (we need to go up to m^4 since we need $f \to +\infty$ as $|m| \to \infty$, so the free energy stays bounded). A free energy of the form $f(m,T) = f_0 + am^2 - bm^4 + cm^6$, with $b < 0$ can be shown to describe a *tricritical* point, where a line of first order phase transitions meets a line of continuous phase transitions; this behavior is observed in an Ising anti-ferromagnet in an external magnetic field. We will return to Landau Theory later.

4) Specific Heat at $h = 0$ and the critical exponent α

Since we are in the ensemble in which h is fixed, if we want to compute the specific heat per spin at $h = 0$ we should use the Gibbs free energy density.

$$s = -\left(\frac{\partial g}{\partial T}\right)_{h=0} \quad \Rightarrow \quad c = T\left(\frac{\partial s}{\partial T}\right)_{h=0} = -T\left(\frac{\partial^2 g}{\partial T^2}\right)_{h=0} \tag{4.4.17}$$

From the previous discussion, $g(0,T) = f(m^*,T)$, where m^* is the value of m that minimizes the $f(m,T)$ of Eq. (4.4.16).

For $T > T_c$, $m^* = 0$, and so $f(m,T) = f_0(T)$ and $c = -T\left(\dfrac{\partial^2 f_0}{\partial T^2}\right)$.

For $T < T_c$, $m^* = \pm m_0 = \pm\sqrt{-a/2b}$, so

$$g(0,T) = f(m^*,T) = f_0(T) + a{m^*}^2 + b{m^*}^4 = f_0(T) - \frac{a^2}{2b} + \frac{ba^2}{4b^2} \tag{4.4.18}$$

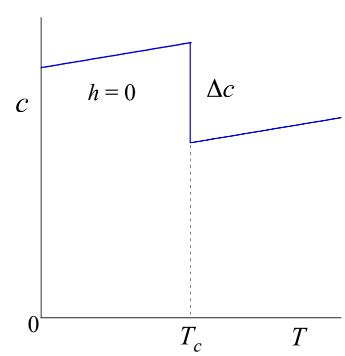
$$= f_0(T) - \frac{a^2}{4b} = f_0(T) - \frac{a_0^2(T-T_c)^2}{4b} \tag{4.4.19}$$

and so, for $T < T_c$,

$$c = -T\left(\frac{\partial^2 f_0}{\partial T^2}\right)_{h=0} + \frac{a_0^2}{2b}T \tag{4.4.20}$$

we thus have,

$$c = \begin{cases} -T\left(\dfrac{\partial^2 f_0}{\partial T^2}\right)_{h=0} + \dfrac{a_0^2}{2b}T & \text{for } T < T_c \\ -T\left(\dfrac{\partial^2 f_0}{\partial T^2}\right)_{h=0} & \text{for } T > T_c \end{cases} \tag{4.4.21}$$



The specific heat thus takes a discontinuous jump downwards at T_c, with

$$\Delta c \equiv c(T \to T_c^-) - c(T \to T_c^+) = \frac{a_0^2}{2b}T_c \tag{4.4.22}$$

The piece $\left(\dfrac{\partial^2 f_0}{\partial T^2}\right)$ is a non-singular part of the specific heat that is smooth and continuous as one passes through the transition at T_c.

One customarily defines the specific heat exponent α by the relation $c \propto |t|^{-\alpha}$. We can rewrite this as,

$$\alpha = -\lim_{t\to 0}\left[\frac{\ln c}{\ln|t|}\right] \tag{4.4.23}$$

For the mean-field calculation, the second definition gives $\alpha = 0$, since $\ln c$ is finite at T_c while $\ln|t| \to \infty$.

Summary

For our mean-field approximate solution to the Ising model, we have found the following critical behavior and exponents:

$$\begin{array}{llll}
T < T_c, h = 0 & m_0(T) \sim |t|^\beta & \beta = 1/2 & \\
T = T_c & m(h) \sim h^{1/\delta} & \delta = 3 & \\
h = 0 & \chi(T) \sim |t|^{-\gamma} & \gamma = 1, & \displaystyle\lim_{t\to 0}\frac{\chi +}{\chi^-} = 2 \quad \text{amplitude ratio} \\
h = 0 & c(T) \sim |t|^{-\alpha} & \alpha = 0 &
\end{array}$$

The values of the exponents in mean-field theory are independent of the dimension d of the system. The dimensionality only somewhat entered our calculation in the form of the coordination number z, which then determined the value of T_c. Note, z depends on the geometry of the lattice of spin sites, and so can have different values for different periodic lattices even in the same dimension d.

From Onsager's exact solution, however, we know that the critical exponents in $d = 2$ dimensions have the values:

$$\beta = 1/8, \qquad \delta = 15, \qquad \gamma = 7/4, \qquad \alpha = 0 \quad \text{but with } c \sim \ln|t| \text{ rather than having a jump} \tag{4.4.24}$$

Clearly the mean-field solution is not capturing all the important physics of the problem! What is mean-field leaving out? We will discuss this point in the last section of this unit.