Unit 4-8: Fluctuations and the Ginzburg Criterion

We have discussed the mean-field solutions to the Ising model, in which the critical exponents do not depend on the system dimensionality. We have shown that the mean-field calculation gives the exact answer for an infinite range Ising model, however we find it does not give the correct answer for the 1D Ising model, for which there is no finite temperature phase transition. By comparison with the exact Onsager solution in 2D, and numerical calculations in 3D, we know that the mean-field critical exponents do not give the correct result in 2D or 3D. So what has gone wrong with the mean-field solution?

Early in our discussion of phase transitions we said that we must be in the thermodynamic limit, with $N \to \infty$ degrees of freedom, to have a singular phase transition. But mean-field theory is essentially a theory with only *one* degree of freedom – the order parameter. The singular behavior in the mean-field theory comes when we "fix" the mean-field solution using the Maxwell construction. But there is no true consideration of the many degrees of freedom which give *fluctuations* around the average value of the order parameter m. It is these fluctuations that are responsible for the failure of mean-field theory at lower dimensions.

For the Ising model we had for the magnetic susceptibility, $\chi = \lim_{h\to 0}\left[\dfrac{\partial m}{\partial h}\right] \to \infty$ at T_c. Now,

$$m = -\left(\frac{\partial g}{\partial h}\right)_T \quad\Rightarrow\quad \chi = \left(\frac{\partial m}{\partial h}\right)_{T,h\to 0} = -\left(\frac{\partial^2 g}{\partial h^2}\right)_{T,h\to 0} = \frac{k_BT}{N}\left(\frac{\partial^2 \ln Z}{\partial h^2}\right)_{T,h\to 0} = \frac{k_BT}{N}\left[\frac{1}{Z}\frac{\partial^2 Z}{\partial h^2} - \left(\frac{1}{Z}\frac{\partial Z}{\partial h}\right)^2\right]_{h\to 0} \tag{4.8.1}$$

With $\mathcal{H} = -J\sum_{\langle ij\rangle} s_i s_j$ just the spin interaction part, we can write,

$$Z = \sum_{\{s_i\}} \mathrm{e}^{-\beta\mathcal{H}+\beta hM} \quad\Rightarrow\quad \frac{\partial Z}{\partial h} = \sum_{\{s_i\}} \mathrm{e}^{-\beta\mathcal{H}+\beta hM}\,\beta M \quad\text{and}\quad \frac{\partial^2 Z}{\partial h^2} = \sum_{\{s_i\}} \mathrm{e}^{-\beta\mathcal{H}+\beta hM}\,(\beta M)^2 \tag{4.8.2}$$

We therefore have,

$$\chi = \frac{k_BT}{N}\beta^2\left[\langle M^2\rangle - \langle M\rangle^2\right] = \frac{1}{k_BT}\frac{\langle M^2\rangle - \langle M\rangle^2}{N} \tag{4.8.3}$$

or, with $m = M/N$, we have,

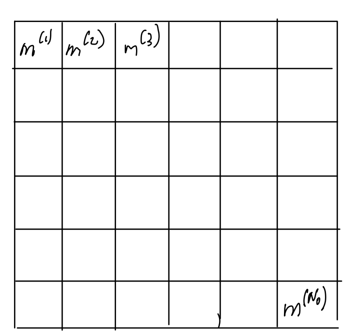
$$\chi = \frac{N}{k_BT}\left[\langle m^2\rangle - \langle m\rangle^2\right] \tag{4.8.4}$$

The divergence of χ at T_c therefore would seem to be related to a divergence in the fluctuations in the magnetization.

For $T \neq T_c$, χ is finite in the thermodynamic limit as $N \to \infty$. Eq. (4.8.4) then implies that $\langle m^2\rangle - \langle m\rangle^2 \sim \dfrac{1}{N}$, or,

$$\sqrt{\langle m^2\rangle - \langle m\rangle^2} \sim \frac{1}{\sqrt{N}} \to 0 \quad \text{as } N \to \infty. \tag{4.8.5}$$

We can understand this $1/\sqrt{N}$ dependence as follows.



Imagine we subdivide our total system into N_0 subsystems. If each subsystem is sufficiently large, we can expect the subsystems will be behaving independently of one another. The measured magnetizations $m^{(i)}$ in each subsystem (i) would then be a set of N_0 independent identically distributed random variables. If the total system average is the average of these $m^{(i)}$, then the variance of m is the variance of the $m^{(i)}$ divided by the number of terms N_0 entering into the average m. The standard deviation of m is then,

$$\sqrt{\langle m^2\rangle - \langle m\rangle^2} \sim \frac{1}{\sqrt{N_0}} \tag{4.8.6}$$

To see this, we can review:

$$\langle m\rangle = \left\langle \frac{1}{N_0}\sum_{i=1}^{N_0} m^{(i)}\right\rangle = \frac{1}{N_0}\sum_{i=1}^{N_0}\langle m^{(i)}\rangle = \langle m^{(i)}\rangle \qquad \text{since all the } m^{(i)} \text{ are identically distributed.} \tag{4.8.7}$$

$$\langle m^2\rangle = \left\langle \left(\frac{1}{N_0}\sum_{i=1}^{N_0} m^{(i)}\right)\left(\frac{1}{N_0}\sum_{j=1}^{N_0} m^{(j)}\right)\right\rangle = \frac{1}{N_0^2}\sum_{i=1}^{N_0}\sum_{j=1}^{N_0}\langle m^{(i)}m^{(j)}\rangle \tag{4.8.8}$$

$$= \frac{1}{N_0^2}\sum_{i=1}^{N_0}\langle [m^{(i)}]^2\rangle + \frac{1}{N_0^2}\sum_{i\neq j}\langle m^{(i)}\rangle\langle m^{(j)}\rangle \qquad \text{since the } m^{(i)} \text{ are independent.} \tag{4.8.9}$$

$$= \frac{1}{N_0}\langle [m^{(i)}]^2\rangle + \frac{N_0(N_0-1)}{N_0^2}\langle m^{(i)}\rangle^2 \qquad \text{since the } m^{(i)} \text{ are identically distributed.} \tag{4.8.10}$$

$$= \frac{1}{N_0}\langle [m^{(i)}]^2\rangle + \left(1-\frac{1}{N_0}\right)\langle m^{(i)}\rangle^2 \tag{4.8.11}$$

So,

$$\langle m^2\rangle - \langle m\rangle^2 = \frac{1}{N_0}\langle [m^{(i)}]^2\rangle + \left(1-\frac{1}{N_0}\right)\langle m^{(i)}\rangle^2 - \langle m^{(i)}\rangle^2 \tag{4.8.12}$$

$$= \frac{\langle [m^{(i)}]^2\rangle - \langle m^{(i)}\rangle^2}{N_0} \sim \frac{1}{N_0} \qquad \Rightarrow \qquad \sqrt{\langle m^2\rangle - \langle m\rangle^2} \sim \frac{1}{\sqrt{N_0}} \tag{4.8.13}$$

Now, as long as the influence of the subsystem at position $\mathbf{r}$ is no longer felt at a finite distance ξ away, one can choose the size of each subsystem to be ξ^d, with d the system dimensionality. So $N_0 = N/\xi^d$, and we conclude,

$$\sqrt{\langle m^2\rangle - \langle m\rangle^2} \sim \sqrt{\frac{\xi^d}{N}} \sim \frac{1}{\sqrt{N}} \tag{4.8.14}$$

When this is the situation, the system is said to be *self averaging*.

But for $T = T_c$, we have $\chi \to \infty$ as $N \to \infty$. So at T_c, $\sqrt{\langle m^2\rangle - \langle m\rangle^2}$ does *not* vanish as quickly as $1/\sqrt{N}$, and the above argument about independent subsystems cannot apply.

This means that the length scale ξ, that describes how far the system is correlated in space, must *diverge* as $T \to T_c$.

For $T \neq T_c$, the state of the system $m(\mathbf{r})$ at position $\mathbf{r}$ has no effect on the sate of the system at $\mathbf{r} + \mathbf{r}_0$, if $\mathbf{r}_0$ is sufficiently large, i.e. $|\mathbf{r}_0| \gg \xi$. But for $T = T_c$, the state of the system at $\mathbf{r}$ influences the state of the system throughout its entire volume, since $\xi \to \infty$.

ξ is called the *correlation length.* ξ diverges at a continuous phase transition. When $\xi = \infty$ we cannot think of the system as a collection of independent subsystems. A thermal fluctuation in $m(\mathbf{r})$ away from its average value m_0 will be felt throughout the entire system. Thus it is crucial to consider fluctuations of the order parameter when trying to compute behavior at such a transition.

Landau-Ginzburg Theory

We have already seen Landau theory in the context of mean-field theory. In Landau theory one defines an order parameter, then expands the Helmholtz free energy as a power series in the order parameter, keeping only the lowest order terms, and using only terms consistent with the symmetry of the system. For the Ising model the order parameter is the magnetization density m, and the Landau free energy density is,

$$f(m,T) = f_0(T) + am^2 + bm^4 \qquad \text{with} \quad a = a_0(T - T_c) \tag{4.8.15}$$

The Gibbs free energy density is then given by,

$$g(h,T) = \min_m \left[f(m,T) - hm\right] \tag{4.8.16}$$

where h is the ordering field conjugate to the order parameter m. The value of m for which the above is minimized gives the equilibrium value of m at the given h.

Now we want to go beyond this mean-field version of Landau theory, to include fluctuations in m away from its average value. This is called the *Landau-Ginzburg* approach. We will assume that the configurations we should consider can be written in terms of small, spatially varying, fluctuations away from the mean field m_0,

$$m(\mathbf{r}) = m_0 + \delta m(\mathbf{r}) \tag{4.8.17}$$

We then construct a free energy *functional* $F[m(\mathbf{r})]$ that tells us the weight of the configuration $m(\mathbf{r})$ in the statistical ensemble,

$$F[m(\mathbf{r})] = \int d^d r \left[f_0 + am^2 + bm^4 + c|\nabla m|^2\right] \tag{4.8.18}$$

The integral is over the d-dimensional space of the system, and the new term in ∇m gives the additional cost in energy to have a configuration with a spatially varying $m(\mathbf{r})$. The $m(\mathbf{r})$ that minimizes F is the constant mean-field solution, $m_0 = \pm\sqrt{-a/2b}$ for $T < T_c$, and $m_0 = 0$ for $T > T_c$.

The goal is now to construct the partition function,

$$Z = \int \mathcal{D}[\delta m(\mathbf{r})]\, \mathrm{e}^{-\beta F[m(\mathbf{r})]} \tag{4.8.19}$$

where the notation $\mathcal{D}[\delta m(\mathbf{r})]$ means we are doing a functional integral over all possible functions $\delta m(\mathbf{r})$. In this way we can explore the effects that small fluctuations have on the mean-field solution.

We start by substituting $m = m_0 + \delta m$ into the free energy functional of Eq. (4.8.18), and expand to $O(\delta m^2)$,

$$F[m(\mathbf{r})] = \int d^d r \left[f_0 + am_0^2 + 2am_0\delta m + a\delta m^2 + bm_0^4 + 4bm_0^3\delta m + 6bm_0^2\delta m^2 + c|\nabla\delta m|^2\right] \tag{4.8.20}$$

The integral over the constant terms, $f_0 + am_0^2 + bm_0^4$, give the mean field free energy F_0.

The linear terms $(2am_0 + 4bm_0^3)\delta\, m$ vanish because $m_0^2 = -a/2b$ minimizes F.

The remaining quadratic terms are,

$$\delta F[\delta m(\mathbf{r})] = \int d^d r \left[\left(a + 6bm_0^2\right)\delta m^2 + c|\nabla\delta m|^2\right] \tag{4.8.21}$$

The integral is over the volume L^d of the system. For convenience, let $\tilde{a} = a + 6bm_0^2$.

To specify the fluctuations $\delta m(\mathbf{r})$, we can express them in terms of Fourier transforms. With periodic boundary conditions, the allowed wavevectors $\mathbf{q}$ satisfy $q_\mu = 2\pi n_\mu/L$, with n_μ integer. We have,

$$\delta m(\mathbf{r}) = \frac{1}{L^{d/2}}\sum_{\mathbf{q}} \mathrm{e}^{i\mathbf{q}\cdot\mathbf{r}}\,\delta m_{\mathbf{q}}, \qquad \delta m_{\mathbf{q}} = \frac{1}{L^{d/2}}\int d^d r\, \mathrm{e}^{-i\mathbf{q}\cdot\mathbf{r}}\delta m(\mathbf{r}) \tag{4.8.22}$$

Then,

$$\delta F = \frac{1}{L^{d/2}}\frac{1}{L^{d/2}}\sum_{\mathbf{q}}\sum_{\mathbf{q}'}\left[\tilde{a} - c\,\mathbf{q}\cdot\mathbf{q}'\right]\delta m_{\mathbf{q}}\delta m_{\mathbf{q}'}\int d^d r\, \mathrm{e}^{i(\mathbf{q}+\mathbf{q}')\cdot\mathbf{r}} \tag{4.8.23}$$

The last integral gives $L^d\delta(\mathbf{q}+\mathbf{q}')$. We then have,

$$\delta F = \sum_{\mathbf{q}}\left[\tilde{a} + cq^2\right]\delta m_{\mathbf{q}}\delta m_{-\mathbf{q}} \tag{4.8.24}$$

Correlation Function

To average over fluctuations, we should compute the partition function averaged over all $\delta m(\mathbf{r})$. With,

$$Z = \int \mathcal{D}[m(\mathbf{r})]\, \mathrm{e}^{-\beta F[m(\mathbf{r})]} = \mathrm{e}^{-\beta F_0} \int \mathcal{D}[\delta m(\mathbf{r})]\, \mathrm{e}^{-\beta \delta F[\delta m(\mathbf{r})]} = \mathrm{e}^{-\beta F_0} \delta Z \tag{4.8.25}$$

the Gibbs free energy at $h = 0$ is then, $G(h = 0, T) = -k_B T \ln Z$.

Now lets transform variables of integration from $\{m(\mathbf{r})\}$ to $\{m_{\mathbf{q}}\}$. Our Fourier transforms were defined so that the Jacobian of this transformation is unity. We have,

$$\delta Z = \left(\prod_{\mathbf{q}} \int d\delta m_{\mathbf{q}} \right) \mathrm{e}^{-\beta \delta F[\delta m_{\mathbf{q}}]} \tag{4.8.26}$$

Note that while $\delta m(\mathbf{r})$ is a real valued function, $\delta m_{\mathbf{q}} = \delta m_{1\mathbf{q}} + i \delta m_{2\mathbf{q}}$ is complex valued. Since $\delta m(\mathbf{r})$ is real, we must have $\delta m^*_{\mathbf{q}} = \delta m_{-\mathbf{q}}$. So $\delta m_{\mathbf{q}}$ and $\delta m_{-\mathbf{q}}$ are not independent. When we integrate over the $\delta m_{\mathbf{q}}$ we should therefore integrate over real values of $\delta m_{1\mathbf{q}}$ and $\delta m_{2\mathbf{q}}$, but restrict $\mathbf{q}$ to $q_z > 0$ so as not to double count $\delta m_{\mathbf{q}}$ and $\delta m_{-\mathbf{q}}$. We thus have,

$$\delta Z = \left(\prod_{\mathbf{q} \text{ s.t.} q_z > 0} \int_{-\infty}^{\infty} d\delta m_{1\mathbf{q}} \int_{-\infty}^{\infty} d\delta m_{2\mathbf{q}} \right) \mathrm{e}^{-\beta \delta F[\delta m_{1\mathbf{q}} + i \delta m_{2\mathbf{q}}]} \tag{4.8.27}$$

We use,

$$\delta F = \sum_{\mathbf{q}} (\tilde{a} + cq^2) \delta m_{\mathbf{q}} \delta m_{-\mathbf{q}} = \sum_{\mathbf{q}} (\tilde{a} + cq^2)(\delta m^2_{1\mathbf{q}} + \delta m^2_{2\mathbf{q}}) = 2 \sum_{\mathbf{q} \text{ s.t.} q_z > 0} (\tilde{a} + cq^2)(\delta m^2_{1\mathbf{q}} + \delta m^2_{2\mathbf{q}}) \tag{4.8.28}$$

where we multiply by 2 since we restrict the last sum to $q_z > 0$, and there is an equal contribution from terms $\mathbf{q}$ and $-\mathbf{q}$.

Now use the fact that the exponential of a sum is equal to the product of exponentials, to write,

$$\delta Z = \prod_{\mathbf{q} \text{ s.t.} q_z > 0} \left[\int_{-\infty}^{\infty} d\delta m_{1\mathbf{q}} \int_{-\infty}^{\infty} d\delta m_{2\mathbf{q}}\, \mathrm{e}^{-2\beta(\tilde{a} + cq^2)(\delta m^2_{1\mathbf{q}} + \delta m^2_{2\mathbf{q}})} \right] \tag{4.8.29}$$

We can now ask, how big is the fluctuation $\delta m_{\mathbf{q}}$ on average?

$$\langle \delta m_{\mathbf{q}} \delta m_{-\mathbf{q}} \rangle = \langle \delta m^2_{1\mathbf{q}} + \delta m^2_{2\mathbf{q}} \rangle \tag{4.8.30}$$

$$= \frac{\displaystyle\int_{-\infty}^{\infty} d\delta m_{1\mathbf{q}} \int_{-\infty}^{\infty} d\delta m_{2\mathbf{q}}\, \mathrm{e}^{-2\beta(\tilde{a} + cq^2)(\delta m^2_{1\mathbf{q}} + \delta m^2_{2\mathbf{q}})}\, (\delta m^2_{1\mathbf{q}} + \delta m^2_{2\mathbf{q}})}{\displaystyle\int_{-\infty}^{\infty} d\delta m_{1\mathbf{q}} \int_{-\infty}^{\infty} d\delta m_{2\mathbf{q}}\, \mathrm{e}^{-2\beta(\tilde{a} + cq^2)(\delta m^2_{1\mathbf{q}} + \delta m^2_{2\mathbf{q}})}} \tag{4.8.31}$$

$$= \langle m^2_{1\mathbf{q}} \rangle + \langle m^2_{2\mathbf{q}} \rangle = \frac{1}{4\beta(\tilde{a} + cq^2)} + \frac{1}{4\beta(\tilde{a} + cq^2)} = \frac{k_B T}{2(\tilde{a} + cq^2)} \tag{4.8.32}$$

We can now compute the spatial correlation function,

$$\langle \delta m(\mathbf{r}) \delta m(0) \rangle = \frac{1}{L^{d/2}} \frac{1}{L^{d/2}} \sum_{\mathbf{q}} \sum_{\mathbf{q}'} \mathrm{e}^{i\mathbf{q}\cdot\mathbf{r}} \langle \delta m_{\mathbf{q}} \delta m_{\mathbf{q}'} \rangle \tag{4.8.33}$$

Because δF involves only $\delta m_{\mathbf{q}} \delta m_{-\mathbf{q}} = \delta m^2_{1\mathbf{q}} + \delta m^2_{2\mathbf{q}}$, then $\langle \delta m_{\mathbf{q}} \delta m_{\mathbf{q}'} \rangle = 0$ unless $\mathbf{q}' = -\mathbf{q}$. We then have,

$$\langle \delta m(\mathbf{r}) \delta m(0) \rangle = \frac{1}{L^d} \sum_{\mathbf{q}} \mathrm{e}^{i\mathbf{q}\cdot\mathbf{r}} \langle \delta m_{\mathbf{q}} \delta m_{-\mathbf{q}} \rangle = \frac{1}{L^d} \sum_{\mathbf{q}} \mathrm{e}^{i\mathbf{q}\cdot\mathbf{r}} \frac{k_B T}{2(\tilde{a} + cq^2)} \tag{4.8.34}$$

Take the $L \to \infty$ limit, where $\sum_{\mathbf{q}} \to \frac{1}{(\Delta q)^d} \int d^d q = \left(\frac{L}{2\pi}\right)^d \int d^d q$, and we get,

$$\langle \delta m(\mathbf{r}) \delta m(0) \rangle = \frac{1}{(2\pi)^d} \int_{-\infty}^{\infty} d^d q \, \mathrm{e}^{i\mathbf{q}\cdot\mathbf{r}} \frac{k_B T}{2(\tilde{a} + cq^2)} \sim \frac{\mathrm{e}^{-r/\xi}}{r^{d-2}} \qquad \text{Ornstein-Zernicke form} \tag{4.8.35}$$

where $\xi = \sqrt{\frac{c}{\tilde{a}}}$ is the *correlation length.* ξ gives the length scale over which fluctuations $\delta m(\mathbf{r})$ decay.

This result for ξ comes from the integral having its poles at $|\mathbf{q}| = \pm i\sqrt{\tilde{a}/c}$.

For $T > T_c$, $\tilde{a} = a + 6bm_0^2 = a = a_0(T - T_c)$, since $m_0 = 0$. This then gives,

$$\xi^+ = \sqrt{\frac{c}{\tilde{a}}} = \frac{\sqrt{c/a_0}}{\sqrt{T - T_c}} \sim \frac{1}{|t|^\nu} \quad \text{with } \nu = 1/2 \tag{4.8.36}$$

For $T < T_c$, $\tilde{a} = a + 6bm_0^2 = a - 6b\left(\frac{a}{2b}\right) = -2a = 2a_0(T_c - T)$. This give,

$$\xi^- = \sqrt{\frac{c}{\tilde{a}}} \sim \frac{\sqrt{c/2a_0}}{\sqrt{T_c - T}} \sim \frac{1}{|t|^\nu} \quad \text{with } \nu = 1/2 \tag{4.8.37}$$

Note, the above gives the amplitude ration $\xi^+/\xi^- = \sqrt{2}$ as $T \to T_c$.

We see that as $T \to T_c$ the correlation length ξ diverges. Since fluctuations propagate out a distance $\xi \to \infty$, one can never divide the system up into independent boxes on any *finite* length scale. This is why $\sqrt{\langle m^2 \rangle - \langle m \rangle^2}$ does not vanish as $1/\sqrt{N}$ at $T = T_c$. This tells us that fluctuations can be important at the transition!

The Ginzburg Criterion

Now we will compute the contribution of the fluctuations to the total Helmholtz free energy.

$$\delta F = \sum_{\mathbf{q}} (\tilde{a} + cq^2) \delta m_{\mathbf{q}} \delta m_{-\mathbf{q}} = 2 \sum_{\mathbf{q} \text{ s.t.} q_z > 0} (\tilde{a} + cq^2)(\delta m_{1\mathbf{q}}^2 + \delta m_{2\mathbf{q}}^2) \tag{4.8.38}$$

$$\delta Z = \prod_{\mathbf{q} \text{ s.t.} q_z > 0} \left[\int_{-\infty}^{\infty} d\delta m_{1\mathbf{q}} \int_{-\infty}^{\infty} d\delta m_{2\mathbf{q}} \, \mathrm{e}^{-2\beta(\tilde{a} + cq^2)(\delta m_{1\mathbf{q}}^2 + \delta m_{2\mathbf{q}}^2)} \right] \tag{4.8.39}$$

$$= \prod_{\mathbf{q} \text{ s.t.} q_z > 0} \left[\frac{2\pi}{4\beta(\tilde{a} + cq^2)} \right]^{1/2} \left[\frac{2\pi}{4\beta(\tilde{a} + cq^2)} \right]^{1/2} \tag{4.8.40}$$

where the first factor comes from the integration over $\delta m_{1\mathbf{q}}$, and the second factor comes from the integration over $\delta m_{2\mathbf{q}}$.

$$\delta Z = \prod_{\mathbf{q} \text{ s.t.} q_z > 0} \left[\frac{\pi k_B T}{2(\tilde{a} + cq^2)} \right] \tag{4.8.41}$$

The contribution to the Gibbs free energy is then,

$$\delta G = -k_B T \ln \delta Z = -k_B T \sum_{\mathbf{q} \text{ s.t.} q_z > 0} \ln \left[\frac{\pi k_B T}{2(\tilde{a} + cq^2)} \right] = -\frac{k_B T}{2} \sum_{\mathbf{q}} \ln \left[\frac{\pi k_B T}{2(\tilde{a} + cq^2)} \right] \tag{4.8.42}$$

where in the last expression we some over all $\mathbf{q}$ rather than just those with $q_z > 0$, and to compensate for that we multiply by 1/2.

We can now, for $L \to \infty$, approximate the sum by an integral to get,

$$\delta G = -\frac{k_B T}{2} \frac{L^d}{(2\pi)^d} \int d^d q \, \ln\left[\frac{\pi k_B T}{2(\tilde{a} + cq^2)}\right] \tag{4.8.43}$$

And the contribution of this to the specific heat per unit volume, δc, will then be,

$$\delta c = -\frac{T}{L^d} \left(\frac{\partial^2 \delta G}{\partial T^2}\right) \tag{4.8.44}$$

Consider $T > T_c$ so that $\tilde{a} = a_0(T - T_c)$ (the result will be similar for $T < T_c$ where $\tilde{a} = 2a_0(T_c - T)$), We have,

$$\frac{1}{L^d}\left(\frac{\partial \delta G}{\partial T}\right) = -\frac{k_B}{2} \frac{1}{(2\pi)^d} \int d^d q \ln\left[\frac{\pi k_B T}{2(\tilde{a} + cq^2)}\right] - \frac{k_B T}{2} \frac{1}{(2\pi)^d} \int d^d q \left[\frac{1}{T} - \frac{a_0}{\tilde{a} + cq^2}\right] \tag{4.8.45}$$

where the second piece of the last term comes from the T dependence of $\tilde{a} = a_0(T - T_c)$.

Then,

$$\frac{1}{L^d}\left(\frac{\partial^2 \delta G}{\partial T^2}\right) = -\frac{k_B}{2} \frac{1}{(2\pi)^d} \int d^d q \left[\frac{1}{T} - \frac{a_0}{\tilde{a} + cq^2}\right] + \frac{k_B}{2} \frac{1}{(2\pi)^d} \int d^d q \, \frac{a_0}{\tilde{a} + cq^2} - \frac{k_B T}{2} \frac{1}{(2\pi)^d} \int d^d q \, \frac{a_0^2}{(\tilde{a} + cq^2)^2} \tag{4.8.46}$$

So,

$$\delta C = \frac{k_B}{2} \frac{1}{(2\pi)^d} \int d^d q \left[1 - \frac{2Ta_0}{\tilde{a} + cq^2} + \frac{T^2 a_0^2}{(\tilde{a} + cq^2)^2}\right] \tag{4.8.47}$$

The first term of "1" gives the classical $\frac{1}{2}k_B$ per harmonic degree of freedom. The second two terms arise from the T-dependence of $a(T) = a_0(T - T_c)$ in δF.

To see how the integrals behave as $T \to T_c$, we have,

$$I_1 = \int d^d q \, \frac{a_0}{a_0 t + cq^2} \qquad \text{where } t = T - T_c. \tag{4.8.48}$$

Let $q^2 = tq'^2$, so that,

$$I_1 = t^{d/2} \int d^d q' \, \frac{a_0}{a_0 t + ctq'^2} = t^{\frac{d}{2}-1} \int d^d q\prime \, \frac{a_0}{a_0 + cq'^2} \tag{4.8.49}$$

The integral is just some number, so we conclude,

$$I_1 \sim t^{\frac{d}{2}-1} = t^{\frac{d-2}{2}} \sim \xi^{2-d} \qquad \text{since } \xi \sim t^{-1/2} \tag{4.8.50}$$

Similarly,

$$I_2 = \int d^d q \, \frac{a_0^2}{(a_0 t + cq^2)^2} \sim t^{\frac{d}{2}-2} = t^{\frac{d-4}{2}} \sim \xi^{4-d} \tag{4.8.51}$$

The second integral I_2 is the more singular one, since it involves a higher power of ξ.

For mean-field theory to be valid, as $T \to T_c$, we want the correction δc to be small compared to the mean-field value c_{MF}.

Recall, in mean-field theory, c_{MF} is finite at T_c, while, from the above, the most singular contribution from the fluctuations gives $\delta c \sim t^{\frac{d-4}{2}}$.

δc will diverge as $T \to T_c$ whenever $d < 4$. We therefore conclude,

$d > 4 \quad \Rightarrow \quad$ fluctuations are negligible, $\delta c \to 0$ as $T \to T_c$, and so mean-field theory will give the correct critical exponents.

$d < 4 \quad \Rightarrow$ fluctuations give singular corrections to the mean-field results; mean-field theory breaks down. The *Renormalization Group* approach gives the way to compute the correct critical exponents, including the effect of fluctuations.

$d_c = 4$ is called the *upper critical dimension.* The value of d_c can vary with the symmetry of the order parameter. For spherically symmetric n-component spin models, including the Ising model, $d_c = 4$.

The condition that determines the value of d_c, as in the above calculation, is called the *Ginzburg criterion.*

There is also a *lower critical dimension*, which depends on the number of components of the spin n. For $d <$ the lower critical dimension, there is no phase transition at finite temperature. For the Ising model, the lower critical dimension is $d = 2$.