**Hall effect** (1879) - determines the sign of the charges that carry the electric current in a metal.

Electric field $E_x$ applied in $\hat{x}$ direction produces a floating electric current $f_x$ in $\hat{x}$ direction. Magnetic field $H$ in $\hat{z}$ direction exerts a Lorentz force $\vec{F}_L$ on the charges carrying the current $f$. For $\vec{f}$ in $\hat{x}$ direction and $\vec{H}$ in $\hat{z}$ direction, the Lorentz force $\vec{F}_L$ is in the $-\hat{y}$ direction. $\vec{F}_L$ deflects the charge carriers to the side wall of the wire (the shaded wall in the figure) where they build up and create a surface charge density. The surface charge density produces an electric field $E_y$ in $\hat{y}$ direction. For a steady state situation, the force from $E_y$ will exactly cancel out the Lorentz force $\vec{F}_L$. If $W$ is the width of the wire, then measuring the "Hall voltage" $V_y = E_y W$ allows one to determine the sign of the charges that carry the electric current.
If current is carried by negative charges \( -q \), then

\[ j_x = -q m v_x \Rightarrow v_x < 0 \]

\( F_x \) deflects the mobile negative charges carrying the current and negative charges build up on shaded surface.

(Neutrality of system \( \Rightarrow \) absence of negative charge, i.e. positive charge, builds up on opposite surface)

The electric field \( E_y \) is in \(-\hat{y}\) direction and Hall voltage is negative.

If current is carried by positive charges \( +q \), then

\[ j_x = q m v_x \Rightarrow v_x > 0 \]

\( F_x \) deflects the mobile positive charges carrying the current and positive charge builds up on the shaded surface.

The electric field \( E_y \) is in the \(+\hat{y}\) direction and the Hall voltage is positive.
For most (but not all) metals one finds a negative Hall voltage. This established that it was negatively charged electrons that carry the electric current in most metals.

**Quantities to measure**

**Hall coefficient** \( \mathcal{R}_H = \frac{E_y}{j_x H} \)

since we expect force from \( E_y \) to exactly balance out Lorentz force \( F_z \) in steady state, we expect \( \mathcal{R}_H \) to be independent of \( H \)

**Magnetoresistance** \( \rho(H) = \frac{E_x}{j_x} \)

We can compute both \( \mathcal{R}_H \) and \( \rho \) using the Drude model.

\[
\frac{d\vec{p}}{dt} = -e \left( \vec{E} + \frac{\vec{p} \times \vec{H}}{mc} \right) - \frac{\vec{p}}{\tau} = 0 \quad \text{in steady state}
\]

for \( x \) and \( y \) components

\[0 = -eE_x - \frac{eH}{mc} p_y - \frac{p_x}{\tau}\]

\[0 = -eE_y + \frac{eH}{mc} p_x - \frac{p_y}{\tau}\]

\[\omega_c = \frac{eH}{mc} \quad \text{cyclotron frequency} = \text{angular frequency of a charged particle in circular motion in uniform} \ H\]
\[ e E_x = -\omega_e p_y - \frac{p_y}{c} \]

\[ e E_y = \omega_e p_x - \frac{p_y}{c} \]

In steady state, current flows only in \( x \) direction. No current flows out of the side walls of the wire \( \Rightarrow p_y = 0 \)

with \( p_y = 0 \),

\[ \Rightarrow p_x = -e E_x \frac{2}{c} \]

\[ j_x = -\frac{m e V_x}{m} = -\frac{m e p_x}{m} = \frac{m e^2}{m} E_x \]

\[ \frac{E_x}{j_x} = \frac{m}{m e^2 c} = \rho \]

Magnetoresistance \( \rho(H) = \frac{1}{\sigma} = \frac{m}{m e^2 c} \)

same as ordinary d.c. resistivity \( \rho \) when \( H = 0 \)

In Drude model, \( \rho(H) \) is independent of \( H \)

Agreed with early measurements by Drude

More modern experiments however do find \( \rho \) can vary with \( H \)

\[ \Rightarrow E_y = \frac{\omega_e}{e} p_x = -\omega_e \frac{2}{c} E_x \]

Hall coefficient \( R_H = \frac{E_y}{j_x H} = \frac{(\frac{\omega_e}{e} p_x)}{(-\frac{m e p_x}{m}) H} = -\omega_e m \)

\[ \frac{e}{m_c} \omega_c = \frac{e H}{m_c} \Rightarrow R_H = -\left( \frac{e}{m_c} \right) \frac{m}{me^2 c H} = -\frac{1}{mec} \]
Hall coefficient, $R_H = -\frac{1}{mc}$

But also, $R_H$ is independent of our phenomenological parameter $\tau$, the relaxation time.

$R_H$ is something we can directly test against experiment since it only depends on the electron density $n$, which can be easily calculated.

In practice, $R_H$ is found to depend on $H$ and $T$ and also on sample preparation. But at low $T$, high $H$ ($\approx 10^4 \text{Gauss}$), very pure samples, $R_H$ is found to approach a constant value, often very close to the Drude value.

<table>
<thead>
<tr>
<th>metal</th>
<th>valence</th>
<th>$-\frac{1}{R_H \cdot m_e c}$</th>
<th>(=) for Drude prediction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Li</td>
<td>1</td>
<td>0.8</td>
<td></td>
</tr>
<tr>
<td>Na</td>
<td>1</td>
<td>1.2</td>
<td></td>
</tr>
<tr>
<td>K</td>
<td>1</td>
<td>1.1</td>
<td></td>
</tr>
<tr>
<td>Rb</td>
<td>1</td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td>Cs</td>
<td>1</td>
<td>0.9</td>
<td></td>
</tr>
<tr>
<td>Cu</td>
<td>1</td>
<td>1.5</td>
<td></td>
</tr>
<tr>
<td>Ag</td>
<td>1</td>
<td>1.3</td>
<td></td>
</tr>
<tr>
<td>Au</td>
<td>1</td>
<td>1.5</td>
<td></td>
</tr>
<tr>
<td>Be</td>
<td>2</td>
<td>-0.2</td>
<td></td>
</tr>
<tr>
<td>Mg</td>
<td>2</td>
<td>-0.4</td>
<td></td>
</tr>
<tr>
<td>In</td>
<td>3</td>
<td>-0.3</td>
<td></td>
</tr>
<tr>
<td>Al</td>
<td>3</td>
<td>-0.3</td>
<td></td>
</tr>
</tbody>
</table>

Drude prediction very good for alkali metals; which drive single $S$ shell electron as valence electron.

Sign is negative! Current is carried by objects with positive sign.
ac electric conductivity

\[ \vec{E}(t) = \text{Re} \left[ \hat{E}_w e^{-i\omega t} \right] \quad \text{harmonic oscillating electric field} \]

\[ \frac{d\vec{p}}{dt} = -\frac{\vec{p}}{c} - e\vec{E}(t) \quad \text{Dyson eqn of motion} \]

assume solution is also harmonic oscillation

\[ \vec{p}(t) = \text{Re} \left[ \hat{p}_w e^{-i\omega t} \right] \]

\[ -i\omega \hat{p}_w = -\hat{p}_w - e\hat{E}_w \]

\[ (\frac{1}{c} - i\omega) \hat{p}_w = -e\hat{E}_w \]

\[ \hat{p}_w = \frac{-e}{\frac{1}{c} - i\omega} \hat{E}_w = \frac{-e\omega}{1-i\omega} \hat{E}_w \]

\[ \text{current} \quad \hat{j}(t) = \text{Re} \left[ \hat{j}_w e^{-i\omega t} \right] \quad \hat{j} = -en\vec{v} \]

\[ \hat{j}_w = -em\hat{p}_w \]

\[ = \frac{me^2c}{m(1-i\omega \tau)} \hat{E}_w \]

a.c. conductivity

\[ \hat{\sigma}_w = \sigma(\omega) \hat{E}_w \]

\[ \Rightarrow \sigma(\omega) = \frac{me^2c}{m(1-i\omega \tau)} = \frac{\sigma_{dc}}{1-i\omega \tau} \]
where $\sigma_{dc} = \frac{me^2c}{m}$ is d.c. Drude conductivity.

as $\omega \to 0$, $\sigma(\omega) \to \sigma_{dc}$

as $\omega \to \infty$, $\sigma(\omega) \to \frac{me^2c}{\omega^2 m} = \frac{ie^2}{m} \text{ indp. of } \omega$

for $\omega \ll 1$, i.e. $\omega \ll \frac{1}{\tau}$, oscillation is fast compared to collision rate, so $\sigma(\omega)$ becomes independent of $\omega$.

Electromagnetic wave propagation in a metal

approx 1) In CGS units, for a propagating electromagnetic plane wave $|E| = |\mathbf{H}|$.

so for the forces the EM wave exerts on a conduction electron

$$\frac{|F_{\text{mag}}|}{|F_{\text{elec}}|} = \frac{-e |\frac{\mathbf{v}}{c} \times \mathbf{H}|}{-e |E|} \approx 1$$

so we will ignore the force that the $\mathbf{H}$ component of the wave exerts on the electron.

approx 2) When wavelength $\lambda$ of EM wave is much longer than mean free path $l$ of collisions, $\lambda \gg l$, the electric field that an electron sees over the time between collisions is roughly uniform in space. Good for waves in visible spectrum where $\lambda \approx 5000 \AA$, $l \approx 10 \AA$.

$(1) + (2) \Rightarrow$ we can use the above computed a.c. conductivity $\sigma(\omega)$ to find the relation.
between the electric field of the EM wave and the resulting current due to the conduction electrons.

For a single harmonic electromagnetic wave we can write for the fields:

\[
\mathbf{E}(\mathbf{r},t) = \text{Re} \left[ E_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \right] \quad \text{electric field}
\]

\[
\mathbf{B}(\mathbf{r},t) = \text{Re} \left[ \mathbf{B}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \right] \quad \text{magnetic field}
\]

The relation between the amplitudes \( E_0 \) and \( \mathbf{B}_0 \) and between the wave vector \( \mathbf{k} \) and frequency \( \omega \) are determined by Maxwell's Equations.

We will look for solutions for a transverse propagating wave, i.e., with \( \mathbf{E}_0 \perp \mathbf{B} \).

Macroscopic Maxwell's Equations (in CGS units)

Gauss \( \nabla \cdot \mathbf{D} = 4\pi \rho \) \quad Gauss \( \nabla \cdot \mathbf{B} = 0 \) \quad Gauss

Ampère \( \nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{j} + \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} \) \quad \( \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \) \quad Faraday

We will ignore magnetization effects, i.e., \( \mathbf{M} = 0 \) and \( \mathbf{B} = \mathbf{H} \).

We will ignore polarization from bound electrons, i.e., \( \mathbf{E} = 0 \) and \( \mathbf{E} = \mathbf{D} \).

\( \mathbf{j} \) is current due to conduction electrons.

\( \mathbf{j} \) is any locally non-neutral charge density due to variations in conduction electron density.
\[ \nabla \cdot \overrightarrow{E} = 0 \]
\[ \nabla \times \overrightarrow{H} = \frac{c}{\eta} \frac{\partial \overrightarrow{E}}{\partial t} + \frac{1}{c} \frac{\partial \overrightarrow{H}}{\partial t} \]
\[ \nabla \times \overrightarrow{E} = -\frac{1}{c} \frac{\partial \overrightarrow{H}}{\partial t} \]
\[ \nabla \cdot \overrightarrow{H} = 0 \]

Substitute into the above the simple harmonic forms for \( \overrightarrow{E} \) and \( \overrightarrow{H} \).

Gauss\( \nabla \cdot \overrightarrow{E} = i k \cdot \overrightarrow{E}_0 \cdot e^{i(k \cdot \overrightarrow{r} - \omega t)} = 0 \) since assumed \( \overrightarrow{E}_0 \parallel \overrightarrow{k} \)

\[ \Rightarrow \overrightarrow{E} = 0 \] transverse EM wave induces no local charge density

Gauss\( \nabla \cdot \overrightarrow{H} = i k \cdot \overrightarrow{H}_0 \cdot e^{i(k \cdot \overrightarrow{r} - \omega t)} = 0 \)

\[ \Rightarrow \overrightarrow{H}_0 \perp \overrightarrow{k} \] so magnetic field is also transverse

1) Faraday
\[ \nabla \times \overrightarrow{E} = -\frac{1}{c} \frac{\partial \overrightarrow{H}}{\partial t} \]
\[ \Rightarrow i k \times \overrightarrow{E}_0 = i \frac{\omega}{c} \overrightarrow{H}_0 \]

2) Ampere\( \nabla \times \overrightarrow{H} = \frac{4\pi}{c} \overrightarrow{j} + \frac{1}{c} \frac{\partial \overrightarrow{E}}{\partial t} \)
\[ \Rightarrow i k \times \overrightarrow{H}_0 = \frac{4\pi}{c} \overrightarrow{j}_0 - \frac{i \omega}{c} \overrightarrow{E}_0 \]

where assumed \( \overrightarrow{j}_0(t) = \text{Re} \left[ \overrightarrow{j}_0 e^{i(k \cdot \overrightarrow{r} - \omega t)} \right] \)

Multiply (1) by \( i \overrightarrow{k} \times \)
\[ i \overrightarrow{k} \times (k \times \overrightarrow{E}_0) = i \overrightarrow{k} (k \cdot \overrightarrow{E}_0) \overrightarrow{k} - i k^2 \overrightarrow{E}_0 = i \frac{\omega}{c} \overrightarrow{k} \times \overrightarrow{H}_0 \]

\[ = 0 \] since \( k \perp \overrightarrow{E}_0 \)

so \( i \overrightarrow{k} \times \overrightarrow{H}_0 = -i \frac{k^2}{\omega} \overrightarrow{E}_0 \)

Substitute from (2)
\[ i \overrightarrow{k} \times \overrightarrow{H}_0 = -i \frac{k^2}{\omega} \overrightarrow{E}_0 = \frac{4\pi}{c} \overrightarrow{j}_0 - \frac{i \omega}{c} \overrightarrow{E}_0 \]
\[ E_0 \left( k^2 - \frac{\omega^2}{c^2} \right) = \frac{4\pi i \omega}{c^2} j_0 \]

For a vacuum, \( j_0 = 0 \) \( \Rightarrow \) \( \omega^2 = c^2 k^2 \)

For a metal \( j_0 = \sigma(\omega) E_0 \), \( \sigma(\omega) \) is ac conductivity

\[ E_0 \left( k^2 - \frac{\omega^2}{c^2} \right) = \frac{4\pi i \omega \sigma(\omega)}{c^2} E_0 \]

\[ E_0 \left( k^2 - \frac{\omega^2}{c^2} - \frac{4\pi i \omega \sigma(\omega)}{c^2} \right) = 0 \]

\[ k^2 = \frac{\omega^2}{c^2} \left( 1 + \frac{4\pi i \sigma(\omega)}{\omega} \right) \]

\[ \sigma(\omega) = \frac{\sigma_{dc}}{1 - i\omega \tau} \quad \sigma_{dc} = \frac{ne^2}{m} \]

For low frequencies, \( \omega \tau \ll 1 \) \( \Rightarrow \) frequency much smaller than collision rate, \( \sigma(\omega) \approx \sigma_{dc} \)

then \[ k^2 = \frac{\omega^2}{c^2} \left( 1 + \frac{4\pi i \sigma_{dc}}{\omega} \right) \]

For fixed real \( \omega \), \( k \) will have a large imaginary part, when \( \omega \) is low enough that \( 4\pi \omega \sigma_{dc}/\omega \gg 1 \) then

\[ k^2 = \frac{\omega^2}{c^2} \frac{4\pi i \sigma_{dc}}{\omega} \]
\[ k = \frac{1}{c} \sqrt{4\pi \sigma_{dc} \omega} \left( \frac{1+i}{\sqrt{2}} \right) \]

\( k \) is a complex number with equal real and imaginary parts.

since \[ E_0 e^{i(k \cdot r - \omega t)} \]

if we write \[ \vec{k} = \vec{k}_1 + i \vec{k}_2 \]

\[ E = \text{Re} \left\{ E_0 e^{i(k \cdot r - \omega t)} \right\} e^{-\vec{k}_2 \cdot \vec{r}} \]

wave decays as it penetrates into metal.

In low freq limit where \( k_1 \ll k_2 \), wave power decays by factor \( \frac{1}{e} \) for every wavelength it penetrates.

More interesting behavior in higher frequency limit, \( \omega \gg 1 \), i.e. frequency large compared to collision rate.

Then \[ \sigma(\omega) \approx \frac{\delta_{dc}}{-i\omega^2} = \frac{m e^2}{\omega^2} \frac{i}{m} = \frac{m e^2}{\omega^2} i \]

\[ k^2 = \frac{\omega^2}{c^2} \left( 1 - \frac{4\pi m e^2}{m \omega^2} \right) \]
call $\omega_p \equiv \sqrt{\frac{4\pi n e^2}{m}}$ the "plasma frequency"

$k^2 = \frac{\omega^2}{c^2} \left(1 - \frac{(\omega_p)^2}{\omega^2}\right)$

dispersion relation is independent of $\omega$

In general $\omega_p \gg \omega$

For freq $\omega > \omega_p$ we have

$k^2 = \frac{\omega^2}{c^2} \left(1 - \frac{(\omega_p)^2}{\omega^2}\right)$ is positive real number

$k = \frac{\omega}{c} \sqrt{1 - \left(\frac{\omega_p}{\omega}\right)^2}$ is real

EM wave propagates through the metal with no attenuation. For $\omega > \omega_p$, the metal is transparent to EM waves!

But for freq $\frac{1}{\omega} \ll \omega < \omega_p$

$k^2 = \frac{\omega^2}{c^2} \left(1 - \frac{(\omega_p)^2}{\omega^2}\right) < 0$ is negative

$k = i \frac{\omega}{c} \sqrt{1 - \left(\frac{\omega_p}{\omega}\right)^2}$ is pure imaginary

$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i (\vec{k} \cdot \vec{r} - \omega t)}$

$\text{Re} \left[ \vec{E}_0 e^{i \omega t} \right] = \text{Re} \left[ \vec{E}_0 e^{i \omega t} \right] e^{-\kappa \kappa_0^2}$

field decays exponentially - waves do not propagate
$\omega < \omega_p$, EM waves get absorbed by the metal.

The crossover from absorption to transparency occurs at $\omega = \omega_p$, or at wavelength

$$\lambda_p = \frac{2\pi c}{\omega_p}$$

$$\omega_p = \sqrt{\frac{4\pi n e^2}{m}}$$

depends only on electron density.

Compare this Drude prediction to experiment.

<table>
<thead>
<tr>
<th>metal</th>
<th>$\lambda_p$ (Drude) ($10^3 \text{Å}$)</th>
<th>$\lambda_p$ (expt) ($10^3 \text{Å}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Li</td>
<td>1.5</td>
<td>2.0</td>
</tr>
<tr>
<td>Na</td>
<td>2.0</td>
<td>2.1</td>
</tr>
<tr>
<td>K</td>
<td>2.8</td>
<td>3.1</td>
</tr>
<tr>
<td>Rb</td>
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<td>3.6</td>
</tr>
<tr>
<td>Cs</td>
<td>3.5</td>
<td>4.4</td>
</tr>
</tbody>
</table>

agreement is not bad given all the simplifying approximations that we have made!