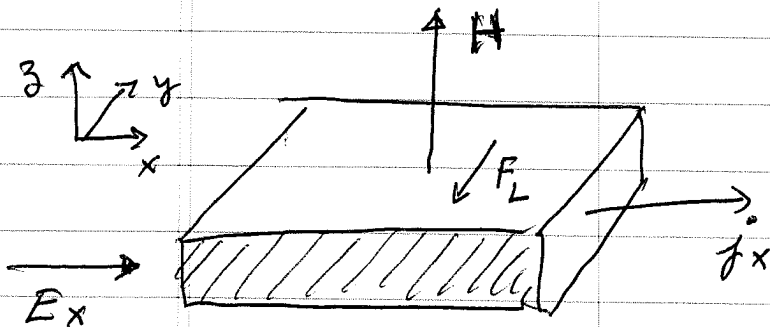


Hall effect (1879) - determines the sign of the charges that carry the electric current in a metal

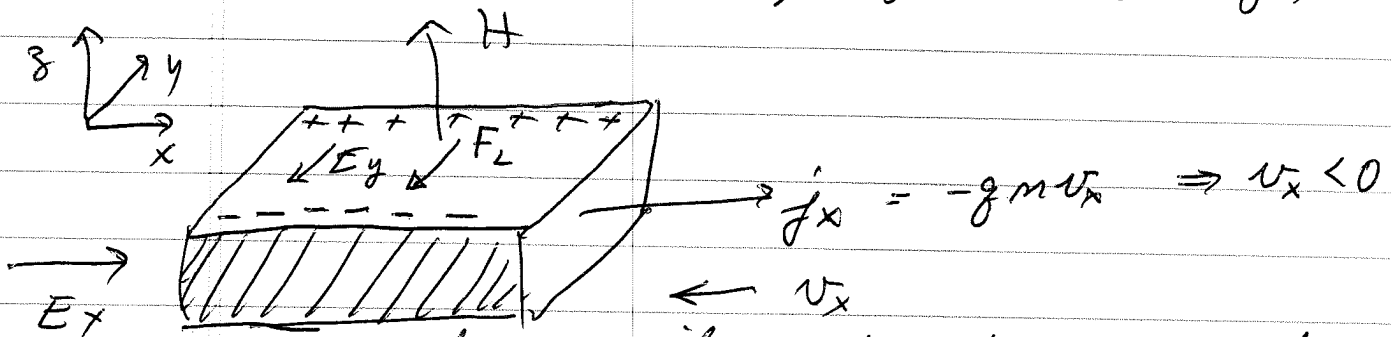
electron motion in combined static electric and magnetic fields



Electric field E_x applied in \hat{x} direction produces flowing electric current j_x in \hat{x} direction. Magnetic field H in \hat{z} direction exerts

Lorentz force $\vec{j} \times \vec{H}$ on the ~~charges~~ moving charges carrying the current \vec{j} . For \vec{j} in \hat{x} direction and \vec{H} in \hat{z} direction, this Lorentz force \vec{F}_L is in the $-\hat{y}$ direction. \vec{F}_L deflects the charge carriers to the side wall of the wire (the shaded wall in the figure) where they build up and create a surface charge density. The surface charge density produces an electric field E_y in \hat{y} direction. For a steady state situation, the force from E_y will exactly cancel out the Lorentz force F_L . If w is the width of the wire, then measuring the "Hall voltage" $V_y = E_y w$ allows one to determine the sign of the charges that carry the electric current.

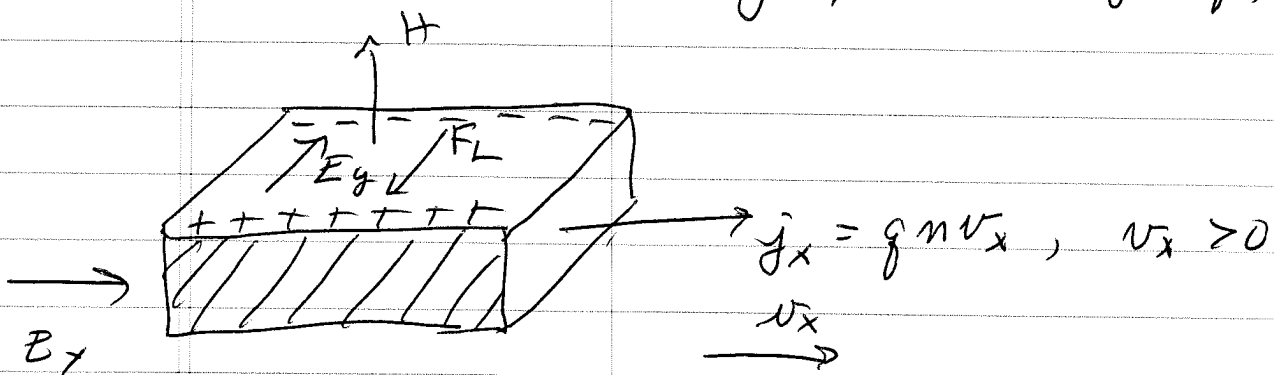
If current is carried by negative charges $-q$, then



F_L deflects the mobile negative charges carrying the current, and negative charges build up on shaded surface (neutrality of system \Rightarrow absence of negative charge, i.e. positive charge, builds up on opposite surface)

The electric field E_y is in $-\hat{y}$ direction and Hall voltage is negative

If current is carried by positive charges $+q$, then



F_L deflects the mobile positive charges carrying the current and positive charge builds up on the shaded surface

The electric field E_y is in the $+\hat{y}$ direction and the Hall voltage is positive

For most (but not all) metals one finds a negative Hall voltage. This established that it was negatively charged electrons that carry the electric current in most metals.

Quantities to measure

Hall coefficient $R_H \equiv \frac{E_y}{j_x H}$

since we expect force from E_y to exactly balance out Lorentz force F_L in steady state, we expect R_H to be independent of H

magnetoresistance $\rho(H) \equiv \frac{E_x}{j_x}$

We can compute both R_H and ρ using the Drude model.

$$\frac{d\vec{p}}{dt} = -e(\vec{E} + \frac{\vec{p}}{mc} \times \vec{H}) - \frac{\vec{p}}{\tau} = 0 \text{ in steady state}$$

for x and y components

$$0 = -eE_x - \frac{eH}{mc} p_y - \frac{p_x}{\tau}$$

$$0 = -eE_y + \frac{eH}{mc} p_x - \frac{p_y}{\tau}$$

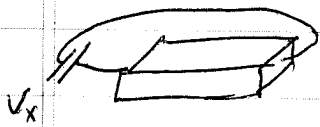
$$\omega_c \equiv \frac{eH}{mc}$$

cyclotron frequency = angular frequency of a charged particle in circular motion in uniform H

$$\textcircled{1} \quad eE_x = -\omega_c p_y - \frac{p_x}{\tau}$$

$$\textcircled{2} \quad eE_y = \omega_c p_x - \frac{p_y}{\tau}$$

In steady state, current flows only in \hat{x} direction.



No current flows out the side walls of the wire $\Rightarrow p_y = 0$

with $p_y = 0$,

$$\textcircled{1} \Rightarrow p_x = -eE_x \tau$$

$$j_x = -mev_x = -\frac{me p_x}{m} = \frac{me^2 e}{m} E_x$$

$$\boxed{\frac{E_x}{j_x} = \frac{m}{me^2 \tau} = \rho}$$

$$\text{magnetoresistance } \rho(H) = \frac{1}{\sigma} = \frac{m}{me^2 \tau}$$

same as ordinary d.c. resistivity ρ when $H=0$

In Drude model, $\rho(H)$ is independent of H !

Agreed with expt measurements by Drude. More modern expts however do find ρ can vary with H .

$$\textcircled{2} \Rightarrow E_y = \frac{\omega_c}{e} p_x = -\omega_c \tau E_x$$

$$\text{Hall coefficient} \quad R_H = \frac{E_y}{j_x H} = \frac{\left(\frac{\omega_c}{e} p_x\right)}{\left(-\frac{me p_x}{m}\right) H} = \frac{-\omega_c m}{me^2 H}$$

$$\text{use } \omega_c = \frac{eH}{mc} \Rightarrow R_H = -\frac{\left(\frac{eH}{mc}\right) m}{me^2 H} = -\frac{1}{mec}$$

$$R_H = -\frac{1}{nec}$$

Hall coefficient independent of H

But also, R_H is independent of our phenomenological parameter τ , the relaxation time.

R_H is something we can directly test against experiment since it only depends on the electron density n , which can be easily calculated.

In practice R_H is found to depend on H and T and also on sample preparation. But at low T , high H ($\sim 10^4$ gauss) very pure samples, R_H is found to approach a constant value, often very close to the Drude value

	metal	valence	$-\frac{1}{R_H n e c}$	(=1 for Drude)
alkali metals	Li	1	0.8	Drude prediction very good for alkali metals which have single s shell electron as valence electron
	Na	1	1.2	
	K	1	1.1	
	Rb	1	1.0	
	Cs	1	0.9	
noble metals filled d-levels	Cu	1	1.5	sign is negative! ⇒ current is carried by objects with positive sign
	Ag	1	1.3	
	Au	1	1.5	
Be	2	-0.2		
Mg	2	-0.4		
In	3	-0.3		
Al	3	-0.3		

a.c. electric conductivity

$$\vec{E}(t) = \text{Re} \left[\vec{E}_\omega e^{-i\omega t} \right] \quad \text{harmonic oscillating electric field}$$

$$\frac{d\vec{p}}{dt} = -\frac{\vec{p}}{\tau} - e\vec{E}(t) \quad \text{Drude eqn of motion}$$

assume solution is also harmonic oscillation

$$\vec{p}(t) = \text{Re} \left[\vec{p}_\omega e^{-i\omega t} \right]$$

$$\Rightarrow -i\omega \vec{p}_\omega = -\frac{\vec{p}_\omega}{\tau} - e\vec{E}_\omega$$

$$\left(\frac{1}{\tau} - i\omega\right) \vec{p}_\omega = -e\vec{E}_\omega$$

$$\vec{p}_\omega = \frac{-e}{\frac{1}{\tau} - i\omega} \vec{E}_\omega = \frac{-e\tau}{1 - i\omega\tau} \vec{E}_\omega$$

$$\text{current } \vec{j}(t) = \text{Re} \left[\vec{j}_\omega e^{-i\omega t} \right] \quad \begin{aligned} \vec{j} &= -en\vec{v} \\ &= -en\frac{\vec{p}}{m} \end{aligned}$$

$$\begin{aligned} \vec{j}_\omega &= -en\frac{\vec{p}_\omega}{m} \\ &= \frac{me^2\tau}{m(1 - i\omega\tau)} \vec{E}_\omega \end{aligned}$$

a.c. conductivity

$$\vec{j}_\omega = \sigma(\omega) \vec{E}_\omega$$

$$\Rightarrow \sigma(\omega) = \frac{me^2\tau}{m(1 - i\omega\tau)} = \frac{\sigma_{dc}}{1 - i\omega\tau}$$

where $\sigma_{dc} = \frac{ne^2\tau}{m}$ is d.c. Drude conductivity

as $\omega \rightarrow 0$, $\sigma(\omega) \rightarrow \sigma_{dc}$

as $\omega \rightarrow \infty$, $\sigma(\omega) \rightarrow \frac{ne^2\tau}{-i\omega\tau m} = \frac{ime^2}{m\omega}$ indep of τ

for $\omega\tau \gg 1$, i.e. $\omega \gg \frac{1}{\tau}$, oscillation is fast compared to collision rate, so $\sigma(\omega)$ becomes independent of τ .

Electromagnetic wave propagation in a metal

approx 1) In CGS units, for a propagating electromagnetic plane wave $|\vec{E}| = |\vec{H}|$.

So for the forces the EM wave exerts on a conduction electron

$$\frac{|\vec{F}_{mag}|}{|\vec{F}_{elec}|} = \frac{-e \left| \frac{\vec{v}}{c} \times \vec{H} \right|}{-e |\vec{E}|} \sim \frac{v}{c} \ll 1$$

so we will ignore the force that the \vec{H} component of the wave exerts on the electron

approx 2) when wavelength λ of EM wave is much longer than mean free path l of collisions, $\lambda \gg l$, the electric field that an electron sees over the time between collisions is roughly ~~constant~~ uniform in space. Good for waves in visible spectrum where $\lambda \sim 5000 \text{ \AA}$, $l \sim 10 \text{ \AA}$.

(1) + (2) \Rightarrow we can use the above computed a.c. conductivity $\sigma(\omega)$ to find the relation

between the electric field of the EM wave and the resulting current due to the conduction electrons

For a single harmonic electromagnetic wave we can write for the fields:

$$\begin{aligned} \vec{E}(\vec{r}, t) &= \text{Re} \left[\vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \right] && \text{electric field} \\ \vec{H}(\vec{r}, t) &= \text{Re} \left[\vec{H}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \right] && \text{magnetic field} \end{aligned}$$

The relation between the amplitudes \vec{E}_0 and \vec{H}_0 and between the wave vector \vec{k} and frequency ω are determined by Maxwell's Equations.

We will look for solutions for a transverse propagating wave, i.e. with $\vec{E}_0 \perp \vec{k}$.

Macroscopic Maxwell's Equations (in CGS units)

$$\begin{aligned} \text{Gauss} \quad \vec{\nabla} \cdot \vec{D} &= 4\pi \rho && \vec{\nabla} \cdot \vec{B} = 0 && \text{Gauss} \\ \text{Ampere} \quad \vec{\nabla} \times \vec{H} &= \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{D}}{\partial t} && \vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} && \text{Faraday} \end{aligned}$$

We will ignore magnetization effects, i.e. $\mu = 1$ and $\vec{B} = \vec{H}$

We will ignore polarization from bound electrons, i.e. $\epsilon = 1$ and $\vec{E} = \vec{D}$

\vec{j} is current due to conduction electrons

ρ is any locally non-neutral charge density due to variations in conduction electron density

Above become

$$\vec{\nabla} \cdot \vec{E} = \rho$$

$$\vec{\nabla} \times \vec{H} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{H}}{\partial t}$$

Substitute into the above the single harmonic forms for \vec{E} and \vec{H} .

Gauss $\vec{\nabla} \cdot \vec{E} = i \vec{k} \cdot \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} = 0$ since assumed $\vec{E}_0 \perp \vec{k}$
 $\Rightarrow \rho = 0$ transverse EM wave induces no local charge density

Gauss $\vec{\nabla} \cdot \vec{H} = i \vec{k} \cdot \vec{H}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} = 0$
 $\Rightarrow \vec{H}_0 \perp \vec{k}$ so magnetic field is also transverse

(1) Faraday $\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{H}}{\partial t} \Rightarrow i \vec{k} \times \vec{E}_0 = i \frac{\omega}{c} \vec{H}_0$

(2) Ampere $\vec{\nabla} \times \vec{H} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \Rightarrow i \vec{k} \times \vec{H}_0 = \frac{4\pi}{c} \vec{j}_0 - \frac{c\omega}{c} \vec{E}_0$

where assumed $\vec{j}(\vec{r}, t) = \text{Re} \left[\vec{j}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \right]$

Multiply (1) by $\vec{k} \times$

$$i \vec{k} \times (\vec{k} \times \vec{E}_0) = i (\underbrace{\vec{k} \cdot \vec{E}_0}_{=0 \text{ since } \vec{k} \perp \vec{E}_0}) \vec{k} - i k^2 \vec{E}_0 = i \frac{\omega}{c} \vec{k} \times \vec{H}_0$$

so $i \vec{k} \times \vec{H}_0 = -i \frac{k^2 c}{\omega} \vec{E}_0$

Substitute from (2)

$$i \vec{k} \times \vec{H}_0 = -i \frac{k^2 c}{\omega} \vec{E}_0 = \frac{4\pi}{c} \vec{j}_0 - \frac{c\omega}{c} \vec{E}_0$$

$$\Rightarrow \vec{E}_0 \left(k^2 - \frac{\omega^2}{c^2} \right) = \frac{4\pi i \omega}{c^2} \vec{j}_0$$

For a vacuum, $\vec{j}_0 = 0 \Rightarrow \omega^2 = c^2 k^2$

For a metal $\vec{j}_0 = \sigma(\omega) \vec{E}_0$ $\sigma(\omega)$ is ac conductivity

$$\Rightarrow \vec{E}_0 \left(k^2 - \frac{\omega^2}{c^2} \right) = \frac{4\pi i \omega}{c^2} \sigma(\omega) \vec{E}_0$$

$$\Rightarrow \vec{E}_0 \left(k^2 - \frac{\omega^2}{c^2} - \frac{4\pi i \omega}{c^2} \sigma(\omega) \right) = 0$$

$$\Rightarrow \boxed{k^2 = \frac{\omega^2}{c^2} \left(1 + \frac{4\pi i \sigma(\omega)}{\omega} \right)}$$

← "dispersion relation" for EM waves in a metal.

$$\sigma(\omega) = \frac{\sigma_{dc}}{1 - i\omega\tau}$$

$$\sigma_{dc} = \frac{ne^2\tau}{m}$$

For low frequencies, $\omega\tau \ll 1$, i.e. frequency much smaller than collision rate, $\sigma(\omega) \approx \sigma_{dc}$

$$\text{then } k^2 = \frac{\omega^2}{c^2} \left(1 + \frac{4\pi i \sigma_{dc}}{\omega} \right)$$

For fixed real ω , k will have a large imaginary part. when ω low enough that $4\pi\sigma_{dc}/\omega \gg 1$ then

$$k^2 = \frac{\omega^2}{c^2} \frac{4\pi i \sigma_{dc}}{\omega}$$

$$k = \frac{1}{c} \sqrt{4\pi\sigma_{dc} \omega} \left(\frac{1+i}{\sqrt{2}} \right)$$

k is complex number with equal real and imaginary parts.

$$\text{since } \vec{E} = \text{Re} \left[\vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} \right]$$

$$\text{if we write } \vec{k} = \vec{k}_1 + i\vec{k}_2$$

$$E = \text{Re} \left\{ \vec{E}_0 e^{i(\vec{k}_1\cdot\vec{r} - \omega t)} \right\} e^{-\vec{k}_2\cdot\vec{r}}$$

wave decays as it propagates into metal

In low freq limit where $k_1 \approx k_2$, wave ~~propagates~~ decays by factor k_2 for every wavelength it penetrates.

More interesting behavior in higher frequency limit, $\omega\tau \gg 1$, i.e. frequency large compared to collision rate.

$$\text{then } \sigma(\omega) \approx \frac{\sigma_{dc}}{-i\omega\tau} = \frac{me^2\tau}{m\omega\tau} i = \frac{me^2}{m\omega} i$$

$$k^2 = \frac{\omega^2}{c^2} \left(1 - \frac{4\pi me^2}{m\omega^2} \right)$$

call $\omega_p \equiv \sqrt{\frac{4\pi m e^2}{m}}$ the "plasma frequency"

$$k^2 = \frac{\omega^2}{c^2} \left(1 - \left(\frac{\omega_p}{\omega}\right)^2\right)$$

dispersion relation is independent of z

In general $\omega_p \gg \frac{c}{z}$

For freq $\omega > \omega_p$ we have

$$k^2 = \frac{\omega^2}{c^2} \left(1 - \left(\frac{\omega_p}{\omega}\right)^2\right) \text{ is positive real number}$$

$$k = \frac{\omega}{c} \sqrt{1 - (\omega_p/\omega)^2} \text{ is real}$$

EM wave propagates through the metal with no attenuation. For $\omega > \omega_p$, the metal is transparent to EM waves!

But for freq $\frac{c}{z} \ll \omega < \omega_p$

$$k^2 = \frac{\omega^2}{c^2} \left(1 - \left(\frac{\omega_p}{\omega}\right)^2\right) < 0 \text{ is negative}$$

$$k = i \frac{\omega}{c} \sqrt{\left|1 - (\omega_p/\omega)^2\right|} \text{ is pure imaginary}$$

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\text{Re} \left[\vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} \right]$$

$$= \text{Re} \left[\vec{E}_0 e^{-i\omega t} \right] e^{-|k|\hat{n}\cdot\vec{r}}$$

field decays exponentially - waves do not propagate

$\omega < \omega_p$, EM waves get totally reflected by the metal.

The crossover from reflecting to transparent occurs at $\omega = \omega_p$, or at wavelength

$$\lambda_p \equiv 2\pi c / \omega_p$$

$$\omega_p = \sqrt{\frac{4\pi n e^2}{m}}$$

depends only on electron density.

Compare this Drude prediction to experiment

<u>metal</u>	<u>λ_p (Drude)</u> (10^3 \AA)	<u>λ_p (expt)</u> (10^3 \AA)
Li	1.5	2.0
Na	2.0	2.1
K	2.8	3.1
Rb	3.1	3.6
Cs	3.5	4.4

agreement is not bad given all the simplifying approximations that we have made!