Combining these two conditions we have the the number of allowed values k_x can take is given by

$$\frac{k_{x\,max}}{\Delta k_x} = \frac{L_y \frac{eH}{\hbar c}}{\left(\frac{2\pi}{L_x}\right)} = L_x L_y \frac{eH}{2\pi \hbar c} = L_x L_y \frac{eH}{hc}$$

$$= \frac{L_x L_y H}{\left(\frac{hc}{e}\right)}$$

to get the number of allowed electron states with energy $\frac{\hbar^2 k_z^2}{2m} + \hbar\omega_c(n+1/2)$ we should multiply above by a factor of 2 for the two possible spin states.

Degeneracy $N = \frac{2 L_x L_y H}{\left(\frac{hc}{e}\right)} = \frac{\Phi}{\left(\frac{hc}{2e}\right)} = \frac{\Phi}{\Phi_0}$

where $\Phi = L_x L_y H$ is the total magnetic flux penetrating the system, and

$\Phi_0 = \frac{hc}{2e}$ has units of magnetic flux and is called the "<u>flux quantum</u>"

$\Phi_0 = 2.07 \times 10^{-7}$ gauss-cm^2

degeneracy is $\frac{\Phi}{\Phi_0}$ = number of flux quanta

Consider now just the motion of the electron in the xy plane. The energy of this motion is

$$\tilde{\varepsilon} = \varepsilon - \frac{\hbar^2 k_z^2}{2m} = \hbar\omega_c(n+1/2) \qquad n = 0, 1, 2, \ldots$$

The states corresponding to a given value of n are called the "n^{th} Landau Level". The n^{th} Landau level has a degeneracy of Φ/Φ_0, or equivalently, the number of electrons per unit area that one can put into a given Landau level is

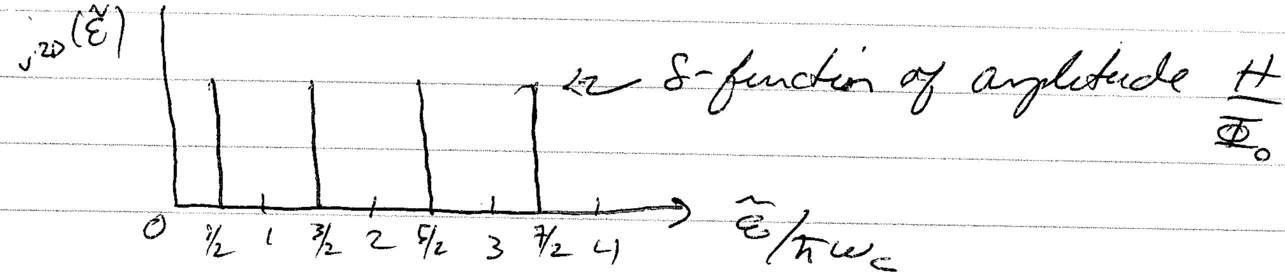
$$\frac{1}{L_x L_y}\frac{\Phi}{\Phi_0} = \frac{H}{\Phi_0}$$

We can summarize this by giving the density of states for the energy $\tilde{\varepsilon}$ in the xy plane

$g_{2D}(\tilde{\varepsilon})\Delta\varepsilon$ = number of electron states per unit area with energy in the range $\tilde{\varepsilon}$ to $\tilde{\varepsilon}+\Delta\varepsilon$

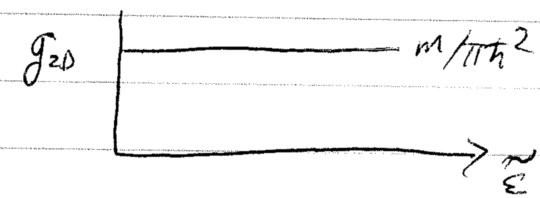
Since there are only states at the discrete energy values $\hbar\omega_c(n+1/2)$, $g_{2D}(\tilde{\varepsilon})$ is a sum of δ-functions at these discrete values - the amplitude of each δ-function is just the degeneracy per area H/Φ_0

$$g_{2D}(\tilde{\varepsilon}) = \sum_n \frac{H}{\Phi_0}\,\delta(\tilde{\varepsilon} - \hbar\omega_c(n+1/2))$$



We can compare this to the 2D density of states when $H=0$. From problem (3b) of HW set 1 you will find that at $H=0$, $g_{2D}(\tilde{\varepsilon})$ is a constant

$$H=0: \quad g_{2D}(\tilde{\varepsilon}) = \frac{m}{\pi\hbar^2}$$



To compare $H=0$ with $H>0$, consider computing the average density of state for $H>0$ where we average over an energy interval large compared to the spacing between the Landau levels $\hbar\omega_c$.

$$\text{average density of states } \bar{g} = \frac{(\#\ \delta\text{-function spikes in } \Delta\varepsilon) \times \frac{H}{\Phi_0}}{\text{interval width } \Delta\varepsilon}$$

If we take $\Delta\varepsilon = M\hbar\omega_c$ for a large integer M, then on average there will be M δ-function spikes in this interval, so

$$\bar{g} = \frac{M \times \frac{H}{\Phi_0}}{M\hbar\omega_c} = \frac{H}{\left(\frac{hc}{2e}\right)} \frac{1}{\hbar\left(\frac{eH}{mc}\right)} = \frac{m}{\pi\hbar^2}$$

so average density of state at $H>0$

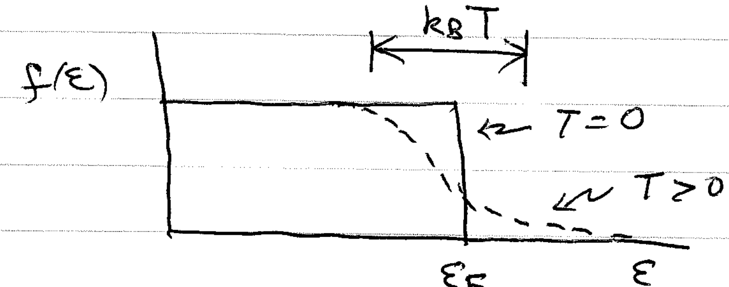
$$\bar{g} = \frac{m}{\pi\hbar^2} = \text{constant density of states at } H=0$$

So turning on the magnetic field bunches the energy eigenstates up into discrete levels, but the average number of states per unit energy remains the same (provided we average on interval $\gg \hbar\omega_c$)

Suppose we had an actual 2D electron gas.
One can think of making this in a thin metallic film or a semiconductor inversion layer where the gas is confined to a region in space along $\hat{z}$ so small that only the lowest allowed value of k_z is occupied, ie $\frac{2\pi}{L_z} = \Delta k_z$ gives $\frac{\hbar^2(\Delta k_z)^2}{2m}$ larger than all other energy scales.

What is necessary so that one could detect the difference between the discrete Landau level structure at finite $H>0$, and the average density of states which is equal to its $H=0$ value?

If f is the Fermi function, we know that finite temperature smears out the sharp cutoff at $\varepsilon = \varepsilon_F$ that exists at $T=0$.



To see the Landau level structure we thus need this smearing to be small on the scale of the spacing between the Landau levels

ie. need $k_B T \ll \hbar\omega_c$

using $\omega_c = \frac{eH}{mc}$ and m the free electron mass one can compute

$$\omega_c = 1.76 \times 10^{11} \text{ sec}^{-1} \quad \text{for a } H = 1 \text{ tesla} = 10^4 \text{ gauss}$$

magnetic field.

1 tesla is a big field. In a laboratory setup such as in BL one can buy a 10 tesla magnet. Larger field strengths require specialized facilities

So for $H = 1$ tesla, $\boxed{\frac{\hbar\omega_c}{k_B} = 1.34\,°K}$

So in a 1 tesla field one needs to go well below 1°K to see Landau level structure. In a 10 tesla field one needs to go well below 10°K. So quite low temperatures are needed.

There is a second condition. In solving Schrodinger's equation for the Landau levels, we ignored any sources of electron scattering (scattering off phonons, plasmons, lattice impurities, etc.) If τ is the scattering time, including such scattering generally leads, via the uncertainty principle, to a broading of the energy levels of the eigenstates to a finite width $\delta E \sim \frac{\hbar}{\tau}$

So to see Landau level structure we need

$$\delta\varepsilon \ll \hbar\omega_c \Rightarrow \frac{\hbar}{\tau} \ll \hbar\omega_c$$

$$\Rightarrow \omega_c\tau \gg 1$$

using $\omega_c = 1.76\times10^{11}\ \mathrm{sec}^{-1}$ in $H = 1$ tesla
and from resistivity measurements used to estimate
τ from Drude's model we get

room temp $\quad \tau \sim 10^{-14}$ sec $\quad , \quad \omega_c\tau \sim 0.00176$
$77^\circ K$ (liquid N) $\quad \tau \sim 10^{-13}$ sec $\quad , \quad \omega_c\tau \sim 0.0176$

we again see that we will need very low temperatures (large τ) to get $\omega_c\tau \gg 1$.

Landau level structure is typically only observable if one goes down to liquid Helium temperatures $\sim 5^\circ K$.

Landau levels and the 3D density of states

We saw $\varepsilon_{k_x,k_y,n} = \frac{\hbar^2 k_z^2}{2m} + \hbar\omega_c(n+1/2)$

$$= \varepsilon_z + \varepsilon_\perp$$

We can write the density of states $g(\varepsilon)$ in 3D as a superposition of the 2D density of states $g_\perp(\varepsilon_\perp)$ for the orbital motion in the xy plane, and the 1D density of states $g_z(\varepsilon_z)$ for the free particle motion along $\hat{z}$

$$g(\varepsilon) = \underset{\substack{\uparrow \\ \text{spin states}}}{2} \int_0^{\varepsilon} d\varepsilon_\perp\, g_\perp(\varepsilon_\perp)\, g_z(\varepsilon - \varepsilon_\perp)$$

For $\varepsilon_z = \frac{\hbar^2 k_z^2}{2m}$, $g_z(\varepsilon_z)\, d\varepsilon_z = \frac{2\, dk_z}{2\pi}$ (for $+k_z$ and $-k_z$; $2\pi \leftarrow \Delta k$)

$$g_z(\varepsilon_z) = \frac{1}{\pi}\frac{dk_z}{d\varepsilon_z} = \frac{1}{2\pi}\sqrt{\frac{2m}{\hbar^2 \varepsilon_z}} \quad \text{for } \varepsilon_z = \frac{\hbar^2 k_z^2}{2m}$$

$$\Rightarrow g(\varepsilon) = \frac{1}{2\pi}\sqrt{\frac{2m}{\hbar^2}} \int_0^{\varepsilon} d\varepsilon_\perp \frac{2 g_\perp(\varepsilon)}{\sqrt{\varepsilon - \varepsilon_\perp}}$$

When $H = 0$, $\varepsilon_\perp = \frac{\hbar^2}{2m}(k_x^2 + k_y^2) \Rightarrow g_\perp(\varepsilon_\perp) = \frac{m}{2\pi\hbar^2}$ constant

$$\Rightarrow \boxed{g_0(\varepsilon) = \frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\varepsilon}}$$

g_0: $H=0$

as we found before

But when $H \neq 0$ then

$$g_\perp(\varepsilon_\perp) = \frac{H}{2\phi_0} \sum_{n=0}^{\infty} \delta\left(\varepsilon_\perp - \hbar\omega_c\left(n+\tfrac{1}{2}\right)\right)$$

the 2 because in $g_\perp(\varepsilon_\perp)$ we do not include the spin degeneracy

$\phi_0 = \frac{\hbar c}{2e}$ is the flux quantum

$\omega_c = \frac{eH}{mc}$ is the cyclotron frequency

Using this $g_\perp(\varepsilon_\perp)$ gives for the 3D density of states

$$\boxed{g(\varepsilon) = \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \hbar\omega_c \sum_{n=0}^{n_{max}} \frac{1}{\sqrt{\varepsilon - \hbar\omega_c(n+1/2)}}}$$

where n_{max} is largest integer so that

$$\hbar\omega_c(n_{max} + 1/2) < \varepsilon$$

Let ε_{F0} be the Fermi energy at $H=0$

Define dimensionless energy as $x = \frac{\varepsilon}{\hbar\omega_c}$

Then

$$g_0(\varepsilon) = \frac{g_0(\varepsilon_{F0})}{\sqrt{x_0}}\sqrt{x} \qquad H=0$$

$x_0 = \frac{\varepsilon_{F0}}{\hbar\omega_c}$

$$g(\varepsilon) = \frac{g_0(\varepsilon_{F0})}{\sqrt{x_0}}\,\frac{1}{2}\sum_{n=0}^{n_{max}} \frac{1}{\sqrt{x-n-\frac{1}{2}}} \qquad H>0$$

$$\bar{g}(\varepsilon) = \frac{g(\varepsilon)}{g_0(\varepsilon)}\sqrt{x_0} \quad \text{then}$$

$$\boxed{\begin{array}{ll} \bar{g}_0(\varepsilon) = \sqrt{x} & H=0 \\ \bar{g}(\varepsilon) = \frac{1}{2}\sum_{n=0}^{n_{max}} \frac{1}{\sqrt{x-n-1/2}} & H>0 \end{array}}$$

Show Fig 1

$\bar{g}(\varepsilon)$ has singularities at the values $x_n = n + 1/2$

These are known as <u>van Hove singularities</u>

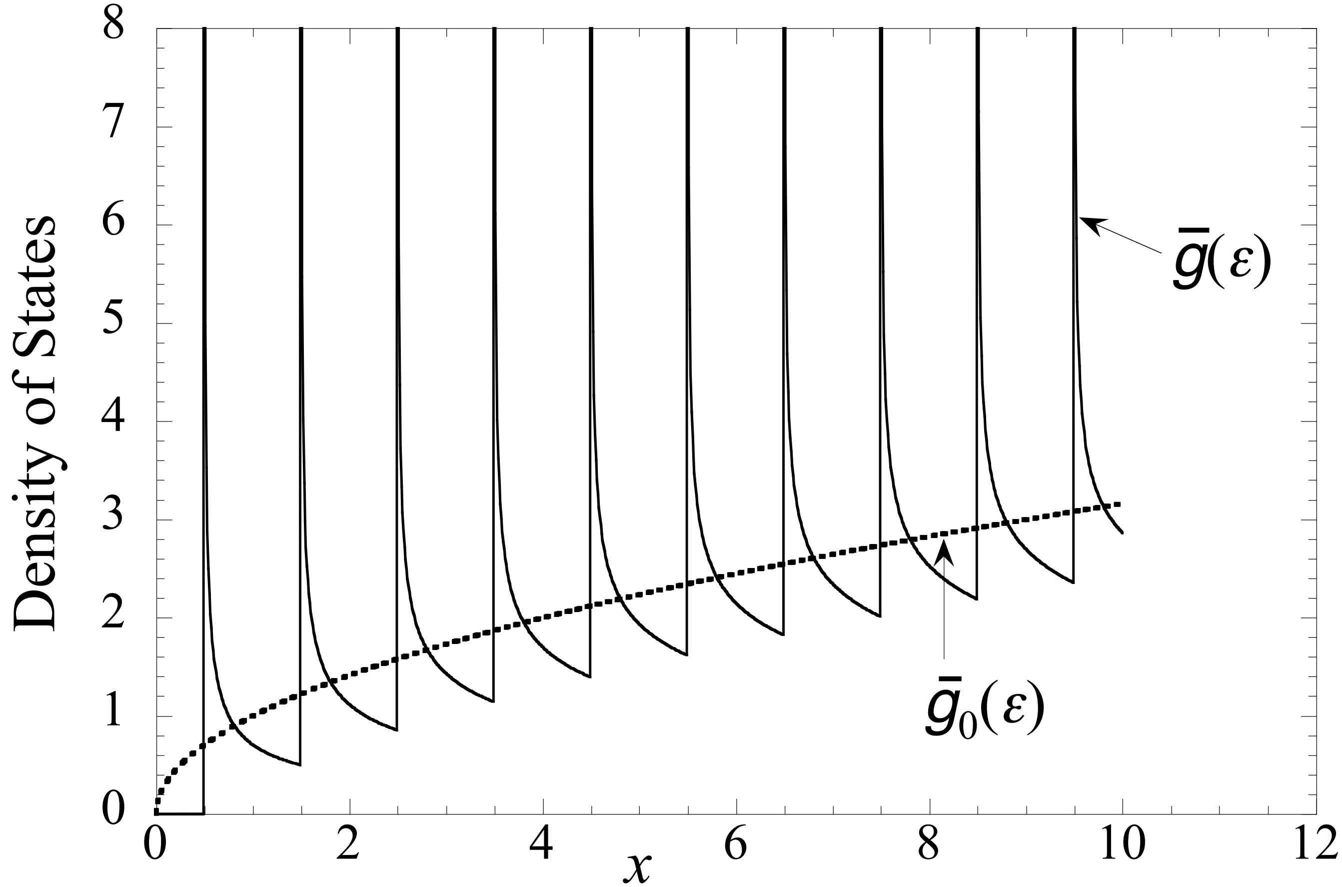
They reflect the discrete nature of the 2D Landau level structure on the 3D density of states

Having found $g(\varepsilon)$ the goal is now to compute the Fermi energy $\varepsilon_F(H)$, then the energy density $u(H)$, and then the magnetization density $M = -\frac{\partial u}{\partial H}$

and then the susceptibility $\chi = \frac{\partial M}{\partial H}$

(In general $M = -\frac{\partial f}{\partial T}$ with f the Helmholtz free energy density. But at $T=0$, $f = u$ the energy density)

Fig. 1



Normalized density of states $\bar{g}_0(\epsilon) = g_0(\epsilon)/c_0$ for zero applied magnetic field (dotted line), and $\bar{g}(\epsilon) = g(\epsilon)/c_0$ for finite applied magnetic field H (solid line), where $c_0 \equiv g_0(\epsilon_{F0})/\sqrt{x_0}$ (see text), vs $x = \epsilon/\hbar\omega_c$, where $\omega_c = eH/mc$ is the cyclotron frequency. In the finite field H, $\bar{g}(\epsilon)$ has van Hove singularities $\sim 1/\sqrt{x - x_n}$ at $x_n = n + 1/2$.

Fermi Energy as function of H

We get ε_F from $\quad n = \int_0^{\varepsilon_F} d\varepsilon\, g(\varepsilon)$

electron density

Useful to consider the integrated density of states

$$G(\varepsilon) \doteq \int_0^{\varepsilon} d\varepsilon\, g(\varepsilon)$$

From our expressions for $g(\varepsilon)$ we get

$$\boxed{\begin{array}{ll} G_0(\varepsilon) = g_0(\varepsilon_{F0})\,\varepsilon_{F0}\,\frac{2}{3}\left(\frac{x}{x_0}\right)^{3/2} & H=0 \\ G(\varepsilon) \approx g_0(\varepsilon_{F0})\,\varepsilon_{F0}\,\frac{1}{(x_0)^{3/2}}\sum_{n=0}^{n_{max}}\sqrt{x-n-\frac{1}{2}} & H>0 \end{array}}$$

When $H=0$, $G_0(\varepsilon_{F0}) = n \Rightarrow g_0(\varepsilon_{F0}) = \frac{3}{2}\frac{n}{\varepsilon_{F0}}$

as we found before

When $H>0$ is turned on, n remains constant but ε_F must shift due to the change in $g(\varepsilon)$

Write

$$\varepsilon_F = \varepsilon_{F0} + \delta\varepsilon$$

$\delta\varepsilon$ is then determined by

$$G(\varepsilon_{F0} + \delta\varepsilon) = G_0(\varepsilon_{F0}) = n$$

Define $\bar{G} = \frac{G}{C_0}$ with $C_0 = \frac{2}{3}\frac{g_0(\varepsilon_{F0})\,\varepsilon_{F0}}{(x_0)^{3/2}}$

So $\boxed{\begin{array}{ll} \bar{G}_0(\varepsilon) = x^{3/2} & H=0 \\ \bar{G}(\varepsilon) = \frac{3}{2}\sum_{n=0}^{n_{max}} \sqrt{x-n-1/2} & H>0 \end{array}}$

Show Fig 2

Then $\bar{G}(\varepsilon_F) = \frac{m}{C_0} = \frac{m \, x_0^{3/2}}{\frac{2}{3} g_0(\varepsilon_{F0}) \varepsilon_{F0}}$

use $g_0(\varepsilon_{F0}) = \frac{3}{2}\frac{m}{\varepsilon_{F0}}$

$$\bar{G}(\varepsilon_F) = \frac{m \, x_0^{3/2}}{\frac{2}{3}\left[\frac{3}{2}\frac{m}{\varepsilon_{F0}}\right]\varepsilon_{F0}} = x_0^{3/2}$$

$$\Rightarrow \frac{\bar{G}(\varepsilon_F)}{x_0^{3/2}} = \boxed{\frac{3}{2}\frac{1}{x_0^{3/2}}\sum_{n=0}^{n_{max}}\sqrt{x_0+\delta x-n-1/2} = 1}$$

$\boxed{n_{max} \text{ is largest integer so that } n_{max}+\tfrac{1}{2} < x_0+\delta x}$

determines shift in Fermi energy

$\delta x = \delta \varepsilon_F$

Solve numerically for δx as function of x_0

$x_0 = \frac{\varepsilon_{F0}}{\hbar \omega_c}$ determined by density m and H

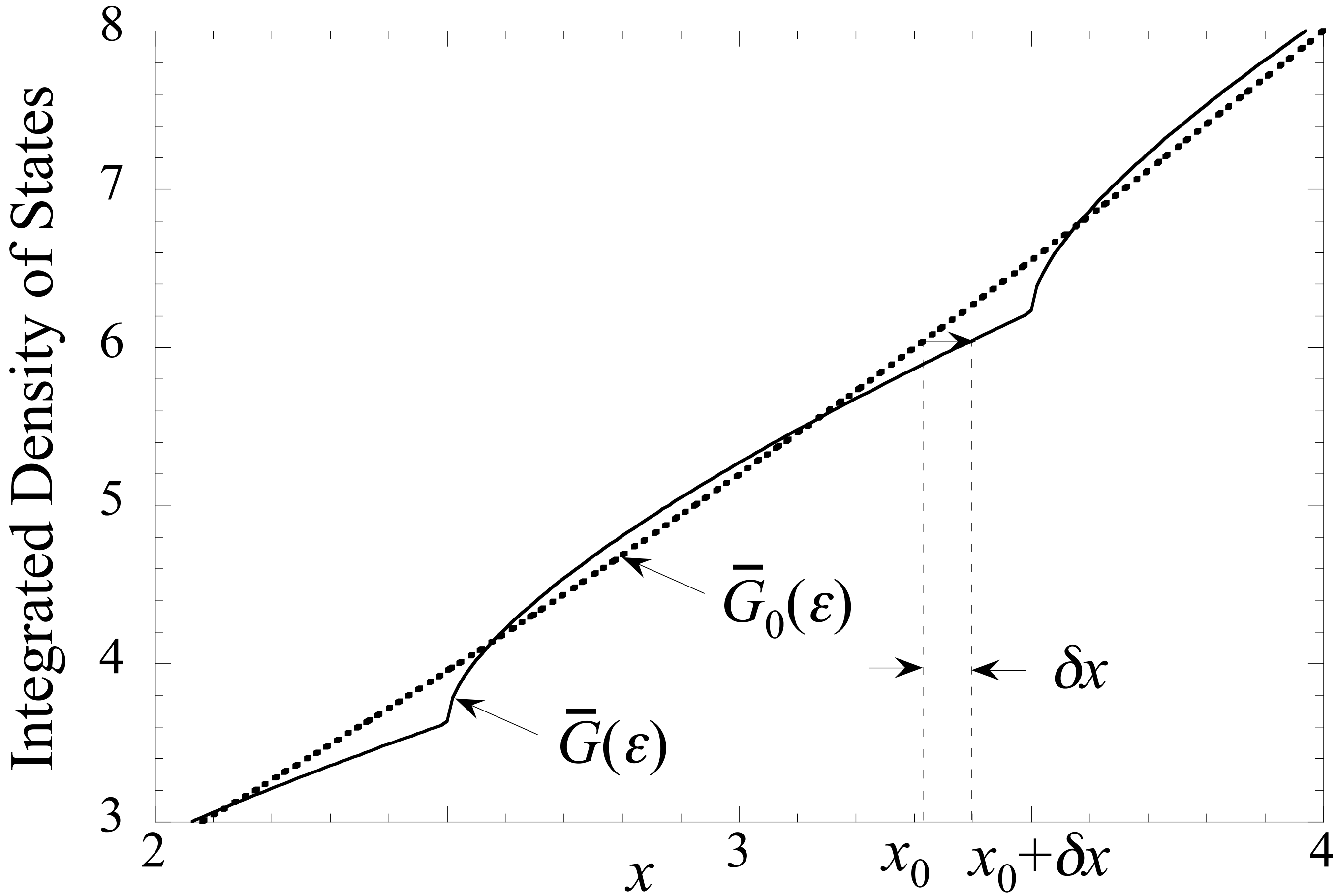
$\delta x = \frac{\varepsilon_F - \varepsilon_{F0}}{\hbar \omega_c}$ shift in Fermi energy

Show Fig 3

δx decreases as x_0 increases

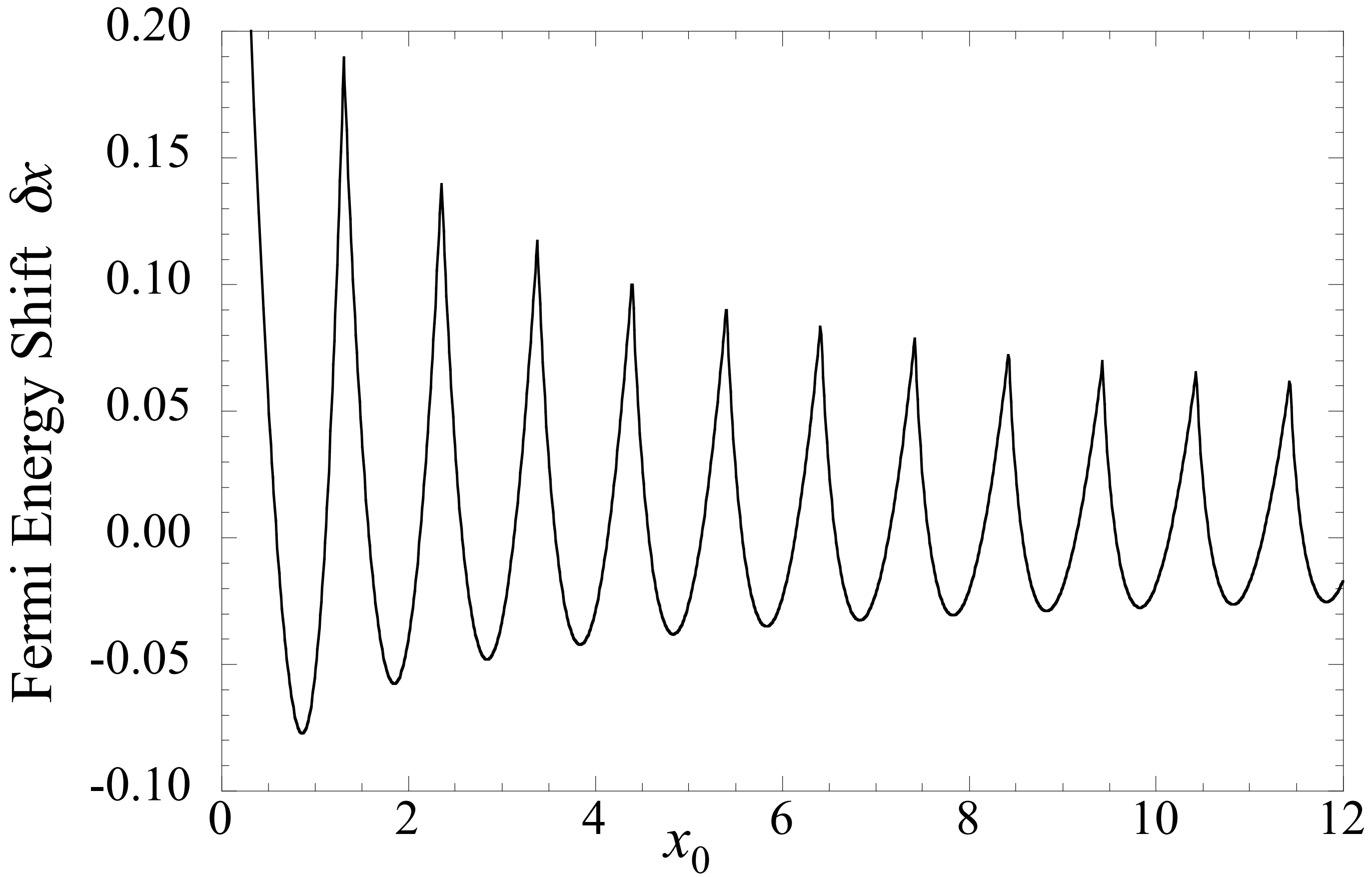
δx oscillates with $\Delta x_0 = 1$

Fig. 2



Normalized integrated density of states $\bar{G}_0(\epsilon) = G_0(\epsilon)/C_0$ for zero applied magnetic field (dotted line), and $\bar{G}(\epsilon) = G(\epsilon)/C_0$ for finite applied magnetic field H (solid line), where $C_0 \equiv (2/3)g_0(\epsilon_{F0})\epsilon_{F0}/(x_0)^{3/2}$ (see text), vs $x = \epsilon/\hbar\omega_c$, where $\omega_c = eH/mc$ is the cyclotron frequency. If x_0 corresponds to the Fermi energy at $H = 0$, the Fermi energy at finite H is given by $x_0 + \delta x$, where δx is determined by $\bar{G}(x_0 + \delta x) = \bar{G}_0(x_0)$, as shown graphically.

Fig. 3



Shift in Fermi energy upon turning on a magnetic field H, $\delta x = \delta\epsilon/\hbar\omega_c$, vs Fermi energy in zero magnetic field $x_0 = \epsilon_{F0}/\hbar\omega_c$, where $\omega_c = eH/mc$ is the cyclotron frequency. δx oscillates with period $\Delta x_0 = 1$.

Ground State Energy

$$u = \int_0^{\varepsilon_F} d\varepsilon\, g(\varepsilon)\,\varepsilon = (\hbar\omega_c)^2 \int_0^{x_F} dx\, g(x)\, x$$

$$\boxed{\begin{aligned} u_0 &= \frac{2}{5}\frac{g_0(\varepsilon_{F0})}{\sqrt{x_0}}(\hbar\omega_c)^2 x_0^{5/2} = \frac{3}{5} n\,\varepsilon_{F0} && H=0 \\ u &= \frac{1}{3}\frac{g_0(\varepsilon_{F0})}{\sqrt{x_0}}(\hbar\omega_c)^2 \sum_{n=0}^{n_{max}} (x_F+2n+1)\sqrt{x_F-n-\tfrac{1}{2}} && H>0 \end{aligned}}$$

$$\frac{u}{u_0} = \frac{5}{6}\frac{1}{x_0^{5/2}}\sum_{n=0}^{n_{max}} (x_0+\delta x+2n+1)\sqrt{x_0+\delta x-n-\tfrac{1}{2}}$$

$$\frac{\varepsilon_F}{\hbar\omega_c} = x_F = x_0+\delta x$$

substitute in for δx from Fig 3

Plot $\frac{\Delta u}{u_0} = \frac{u-u_0}{u_0} = \frac{u}{u_0} - 1$ vs x_0

$\boxed{\text{Show Fig 4}}$

We see $\frac{\Delta u}{u_0} \to 0$ as $x_0 \to \infty$

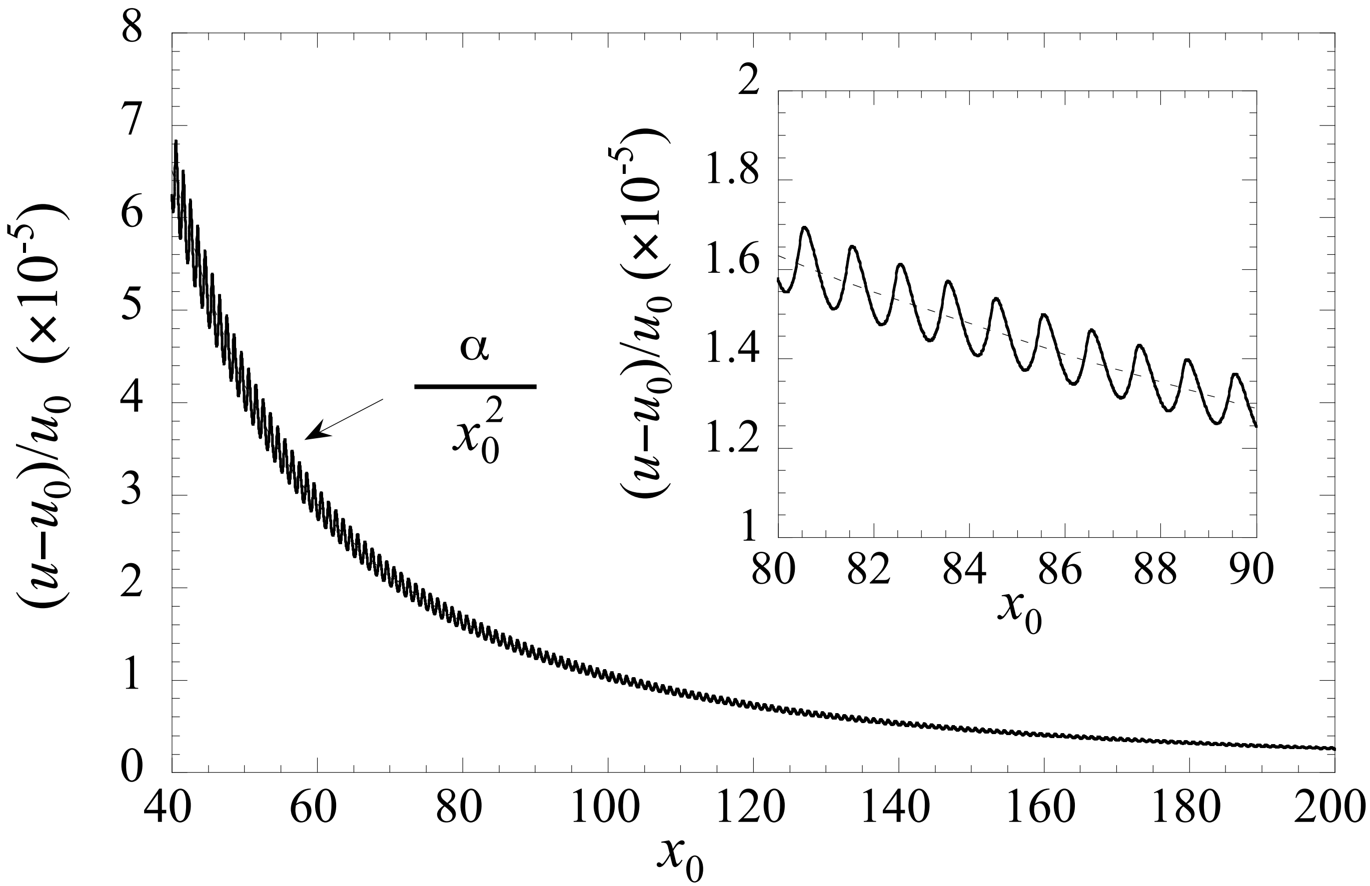
Has to be so since $x_0 = \frac{\varepsilon_{F0}}{\hbar\omega_c} \to \infty$ as $H \to 0$

$\frac{\Delta u}{u_0}$ oscillates with period $\Delta x_0 = 1$

Excellent fit to $u = u_0\left[1 + \frac{\alpha}{x_0^2}\left[1+g(x_0)\right]\right]$

$\alpha = 0.10418$ gives decay gives oscillations

Fig. 4



Relative energy change $(u - u_0)/u_0$ upon turning on a finite magnetic field H vs $x_0 = \epsilon_{F0}/\hbar\omega_c$, where ϵ_{F0} is the Fermi energy for $H = 0$ and $\omega_c = eH/mc$ is the cyclotron frequency. The dashed line is a fit to α/x_0^2 and gives the value $\alpha = 0.10418$. The inset is a blow-up detailing the oscillations with period $\Delta x_0 = 1$.

$\boxed{\text{Show Fig 5}}$

Plot $g(x_0) = \frac{u - u_0}{u_0} \frac{x_0^2}{\alpha} - 1$

$\boxed{\text{Show Fig 6}}$ envelope of $g(x_0)$ is $\frac{\alpha'}{\sqrt{x_0}}$

So

$$u = u_0 + \frac{u_0 \alpha}{x_0^2} + \underbrace{\frac{u_0 \alpha}{x_0^2} g(x_0)}_{}$$

oscillates and $\to 0$ as $x_0 \to \infty$, ie $H \to 0$, can ignore when computing susceptibility at $H \to 0$

using $x_0 = \frac{\varepsilon_{F0}}{\hbar \omega_c}$

$$g_0(\varepsilon_{F0}) = \frac{3}{2} \frac{n}{\varepsilon_{F0}}$$

$$\omega_c = \frac{eH}{mc} \qquad \mu_0 = \frac{e\hbar}{2mc}$$

$$u_0 = \frac{3}{5} n \varepsilon_{F0}$$

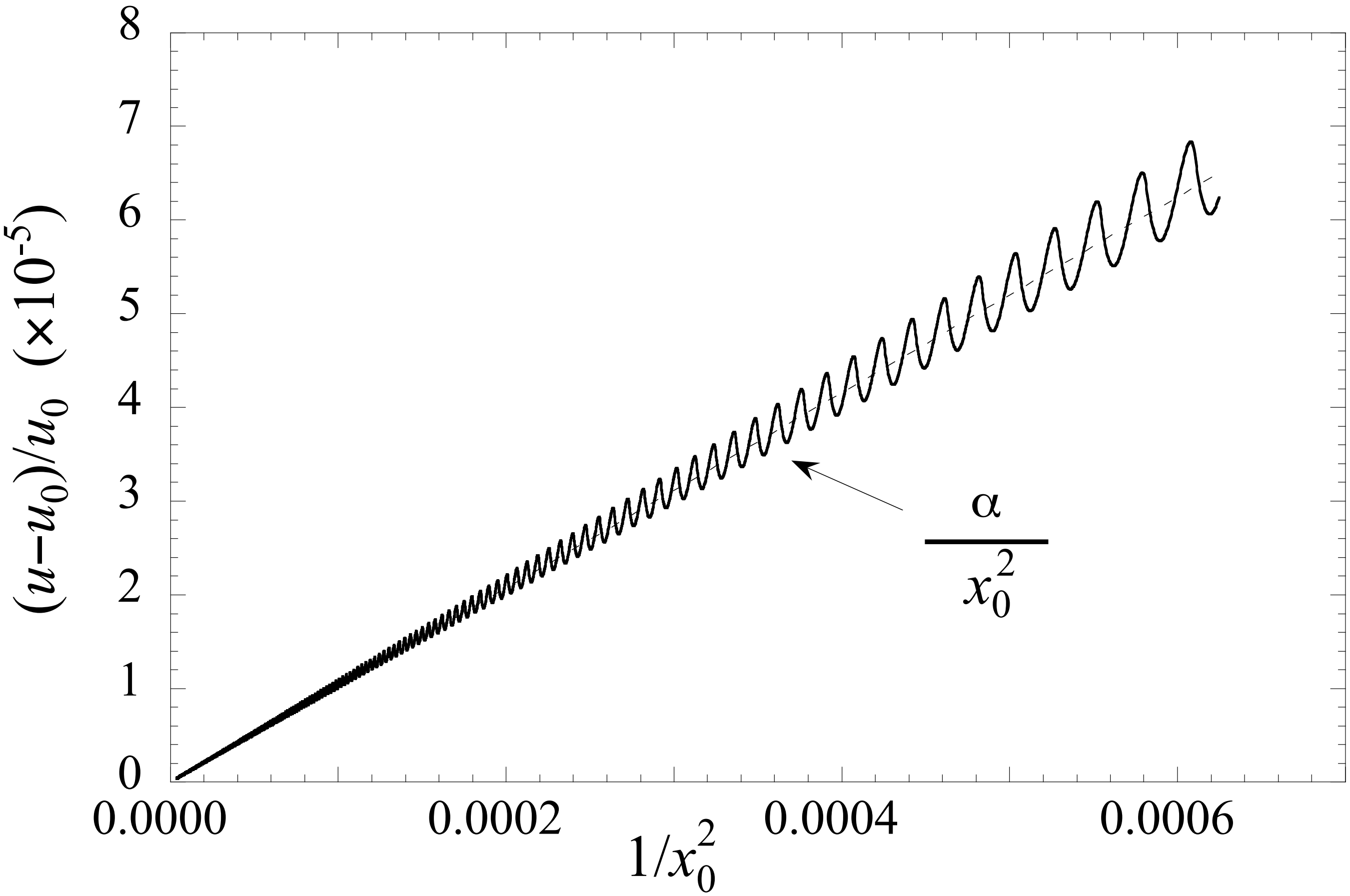
$$u = u_0 + \left(\frac{3}{5} n \varepsilon_{F0}\right) \alpha \left(\frac{\hbar \omega_c}{\varepsilon_{F0}}\right)^2$$

$$= u_0 + \alpha \frac{8}{5} g_0(\varepsilon_F) \mu_0^2 H^2$$

$$\Rightarrow \quad \chi_L = -\frac{\partial u^2}{\partial H^2} = -\alpha \frac{16}{5} g_0(\varepsilon_F) \mu_0^2$$

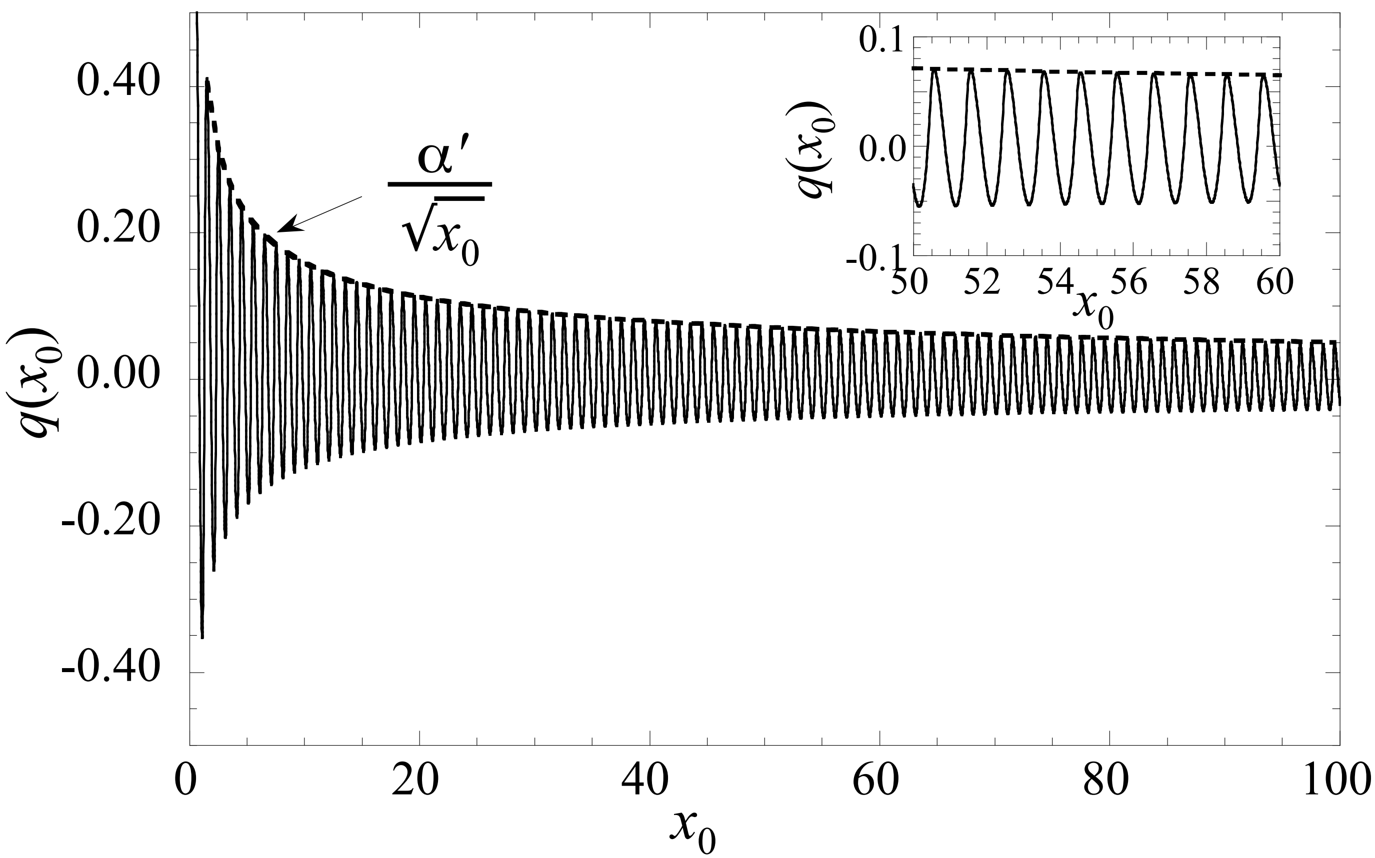
$$\alpha = 0.10418$$

Fig. 5



Relative energy change $(u-u_0)/u_0$ upon turning on a finite magnetic field H vs $x_0 = \epsilon_{F0}/\hbar\omega_c$, where ϵ_{F0} is the Fermi energy for $H = 0$ and $\omega_c = eH/mc$ is the cyclotron frequency. The dashed line is a fit to α/x_0^2 and gives the value $\alpha = 0.10418$. The inset is a blow-up detailing the oscillations with period $\Delta x_0 = 1$.

Fig. 6



Oscillations $q(x_0)$ vs $x_0 = \epsilon_{F0}/\hbar\omega_c$, where ϵ_{F0} is the Fermi energy for $H = 0$ and $\omega_c = eH/mc$ is the cyclotron frequency. The dashed line is a fit of the maxima to the form $\alpha'/\sqrt{x_0}$ and gives the value $\alpha' = 0.50216$. The inset is a blow-up detailing the oscillations with period $\Delta x_0 = 1$.

$$\chi_L = -0.3334\, g_0(\varepsilon_F)\mu_0^2$$

$\uparrow$ diamagnetic

Compare to Landau's analytic calculation at finite T, where he found $\chi_L = -\frac{1}{3} g_0(\varepsilon_F)\mu_0^2$

same result!

Compare to the Pauli paramagnetic susceptibility

$\chi_p = g_0(\varepsilon_{F0})\mu_0^2 \Rightarrow \chi_L = -\frac{1}{3}\chi_p$

Total magnetic susceptibility of electron gas is

$$\chi_{tot} = \chi_L + \chi_p = \frac{2}{3}\chi_p = -2\chi_L$$

net paramagnetic effect.