PHY 521 Solutions Problem Set 1

Hall effect ~~two~~ two species of charge carriers.

1) For the $-e$ charges of mass m and $\vec{p} = m\vec{v}$, the Drude equation of motion is

$$m\frac{d\vec{v}}{dt} = \vec{F} - \frac{m\vec{v}}{\tau} = -e\vec{E} - e\frac{\vec{v}}{c} \times H\hat{z} - \frac{m\vec{v}}{\tau}$$

In steady state $d\vec{v}/dt = 0$. In terms of x, y components

$$\frac{m}{\tau}\begin{pmatrix} v_x \\ v_y \end{pmatrix} = -e\begin{pmatrix} E_x \\ E_y \end{pmatrix} - \frac{eH}{c}\begin{pmatrix} v_y \\ -v_x \end{pmatrix}$$

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} + \tau\left(\frac{eH}{mc}\right)\begin{pmatrix} v_y \\ -v_x \end{pmatrix} = \frac{-e\tau}{m}\begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

using $\omega_c = eH/mc$

$$\begin{pmatrix} v_x + \omega_c\tau v_y \\ v_y - \omega_c\tau v_x \end{pmatrix} = \frac{-e\tau}{m}\begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

or in terms of a matrix

$$\begin{pmatrix} 1 & \omega_c\tau \\ -\omega_c\tau & 1 \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \frac{-e\tau}{m}\begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

current of $-e$ charges is $\vec{j} = -en\vec{v} \Rightarrow \vec{v} = \frac{-\vec{j}}{ne}$

So $$\begin{pmatrix}1 & \omega_c\tau\\ -\omega_c\tau & 1\end{pmatrix}\begin{pmatrix}j_x\\ j_y\end{pmatrix} = \overbrace{\frac{ne^2\tau}{m}}^{\sigma}\begin{pmatrix}E_x\\ E_y\end{pmatrix} \equiv \sigma\begin{pmatrix}E_x\\ E_y\end{pmatrix}$$

For the $+e$ charges use above but
$e \to -e$, and so $\omega_c \to -\omega_c^*$ with $\omega_c^* = \dfrac{eH}{m^* c}$
and $n \to n^*$, $m \to m^*$

$$\begin{pmatrix}1 & -\omega_c^*\tau\\ \omega_c^*\tau & 1\end{pmatrix}\begin{pmatrix}j_x^*\\ j_y^*\end{pmatrix} = \overbrace{\frac{n^* e^2\tau}{m^*}}^{\sigma^*}\begin{pmatrix}E_x\\ E_y\end{pmatrix} \equiv \sigma^*\begin{pmatrix}E_x\\ E_y\end{pmatrix}$$

invert matrices to get

$$\begin{pmatrix}j_x\\ j_y\end{pmatrix} = \frac{\sigma}{1+(\omega_c\tau)^2}\begin{pmatrix}1 & -\omega_c\tau\\ \omega_c\tau & 1\end{pmatrix}\begin{pmatrix}E_x\\ E_y\end{pmatrix}$$

$$\begin{pmatrix}j_x^*\\ j_y^*\end{pmatrix} = \frac{\sigma^*}{1+(\omega_c^*\tau)^2}\begin{pmatrix}1 & \omega_c^*\tau\\ -\omega_c^*\tau & 1\end{pmatrix}\begin{pmatrix}E_x\\ E_y\end{pmatrix}$$

So for the total current $\vec J = \vec j + \vec j^{\,*}$ we get

$$\begin{pmatrix}J_x\\ J_y\end{pmatrix} = \begin{bmatrix}\dfrac{\sigma}{1+(\omega_c\tau)^2} + \dfrac{\sigma^*}{1+(\omega_c^*\tau)^2} & \dfrac{\sigma^*\omega_c^*\tau}{1+(\omega_c^*\tau)^2} - \dfrac{\sigma\omega_c\tau}{1+(\omega_c\tau)^2}\\[2ex] \dfrac{\sigma\omega_c\tau}{1+(\omega_c\tau)^2} - \dfrac{\sigma^*\omega_c^*\tau}{1+(\omega_c^*\tau)^2} & \dfrac{\sigma}{1+(\omega_c\tau)^2} + \dfrac{\sigma^*}{1+(\omega_c^*\tau)^2}\end{bmatrix}\begin{pmatrix}E_x\\ E_y\end{pmatrix}$$

$$\vec J = \overleftrightarrow{\sigma}\,\vec E$$

↑ conductivity tensor

We have $\vec{J} = \overleftrightarrow{\sigma} \cdot \vec{E}$

a) When $H=0$, $\omega_c = \omega_c^* = 0$

$$\begin{pmatrix} J_x \\ J_y \end{pmatrix} = \begin{pmatrix} \sigma+\sigma^* & 0 \\ 0 & \sigma+\sigma^* \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

$$\Rightarrow \vec{J} = (\sigma+\sigma^*)\vec{E}$$

so dc conductivity is just $\boxed{\sigma+\sigma^* = \frac{ne^2\tau}{m} + \frac{n^*e^2\tau}{m^*}}$

b) To find the Hall conductivity and the magnetoresistance we can invert the conductivity tensor to get the resistivity tensor

$$\vec{J} = \overleftrightarrow{\sigma} \cdot \vec{E} \Rightarrow \vec{E} = \overleftrightarrow{\rho} \cdot \vec{J} \quad \text{where } \overleftrightarrow{\rho} = \overleftrightarrow{\sigma}^{-1}$$

Since $\overleftrightarrow{\sigma}$ is a 2×2 matrix, it is easy to invert

$$\overleftrightarrow{\rho} = \frac{1}{\det\overleftrightarrow{\sigma}} \begin{pmatrix} \sigma_{yy} & -\sigma_{xy} \\ -\sigma_{yx} & \sigma_{xx} \end{pmatrix}$$

where $\det\overleftrightarrow{\sigma} = \sigma_{xx}\sigma_{yy} - \sigma_{xy}\sigma_{yx}$

From our expression for $\overleftrightarrow{\sigma}$ we see $\sigma_{xx} = \sigma_{yy}$ and $\sigma_{xy} = -\sigma_{yx}$ so we have

$$\det\overleftrightarrow{\sigma} = \sigma_{xx}^2 + \sigma_{yx}^2$$

For the Hall coefficient we can take current flowing in the $\hat{x}$ direction, so $J_y = 0$

$$\Rightarrow \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{pmatrix} \begin{pmatrix} J_x \\ 0 \end{pmatrix} = \begin{pmatrix} \rho_{xx} J_x \\ \rho_{yx} J_x \end{pmatrix}$$

$$R_H = \frac{E_y}{J_x H} = \frac{\rho_{yx} J_x}{J_x H} = \frac{\rho_{yx}}{H}$$

$$\boxed{R_H = \frac{-\sigma_{yx}}{\det \overleftrightarrow{\sigma}} \frac{1}{H} = \frac{-\sigma_{yx}}{(\sigma_{xx}^2 + \sigma_{yx}^2) H}}$$

If we substitute in our expressions for σ_{xx} and σ_{yx} then clearly R_H becomes a function of magnetic field H. This is unlike the single carrier case where R_H was found to be independent of H in the Drude model.

But if we look in the high field limit, $\omega_c \tau \gg 1$, $\omega_c^* \tau \gg 1$, then

$$\sigma_{xx} = \frac{\sigma}{1+(\omega_c\tau)^2} + \frac{\sigma^*}{1+(\omega_c^*\tau)^2} \simeq \frac{\sigma}{(\omega_c\tau)^2} + \frac{\sigma^*}{(\omega_c^*\tau)^2} \propto \frac{1}{H^2}$$

$$\sigma_{yx} = \frac{\sigma \omega_c \tau}{1+(\omega_c\tau)^2} - \frac{\sigma^* \omega_c^* \tau}{1+(\omega_c^*\tau)^2} \simeq \frac{\sigma}{\omega_c \tau} - \frac{\sigma^*}{\omega_c^* \tau} \propto \frac{1}{H}$$

$$\text{so } R_H \propto \frac{(1/H)}{\left(\frac{1}{H^4} + \frac{1}{H^2}\right) H} \sim \frac{1}{\left(\frac{1}{H^3} + \frac{1}{H}\right) H} \sim \text{indep of } H \text{ as } H \to \infty$$

Since $\sigma_{xx} \sim \frac{1}{H^2}$ while $\sigma_{yx} \sim \frac{1}{H}$ as $H \to \infty$
then $\sigma_{yx} \gg \sigma_{xx}$ and we have

$$R_H \simeq \frac{-\sigma_{yx}}{(\sigma_{yx})^2}\frac{1}{H} = -\frac{1}{\sigma_{yx} H} = \frac{-1}{\left(\frac{\sigma}{\omega_c\tau} - \frac{\sigma^*}{\omega_c^*\tau}\right) H}$$

use $\sigma = \frac{ne^2\tau}{m}$

$\omega_c\tau = \frac{eH\tau}{mc}$

$$= \frac{-1}{\left[\left(\frac{ne^2\tau}{m}\right)\left(\frac{mc}{eH\tau}\right) - \left(\frac{n^*e^2\tau}{m^*}\right)\left(\frac{m^*c}{eH\tau}\right)\right] H}$$

$$\boxed{R_H = \frac{-1}{(n-n^*)ec} \quad \text{as } H \to \infty}$$

$$R_H = \frac{-1}{n_{eff}\, ec} \quad \text{where } n_{eff} = n - n^*$$

If $n = n^*$ then R_H gets large!

c) The magneto resistance is given by

$$\rho = \frac{E_x}{J_x} = \frac{\rho_{xx} J_x}{J_x} = \rho_{xx} = \frac{\sigma_{yy}}{\det \overleftrightarrow{\sigma}}$$

$$= \frac{\sigma_{yy}}{\sigma_{xx}^2 + \sigma_{yx}^2}$$

again, like R_H, ρ will in general depend on the magnetic field H.

However in the large field limit where $\omega_c\tau \gg 1$, $\omega_c^*\tau \gg 1$
we have again $\sigma_{yx} \gg \sigma_{xx}$ and so

$$\rho = \frac{\sigma_{yy}}{\sigma_{yx}^2} \simeq \frac{\frac{\sigma}{(\omega_c\tau)^2} + \frac{\sigma^*}{(\omega_c^*\tau)^2}}{\left(\frac{\sigma}{\omega_c\tau} - \frac{\sigma^*}{\omega_c^*\tau}\right)^2}$$

$$\rho = \frac{\sigma + \sigma^*\left(\frac{\omega_c}{\omega_c^*}\right)^2}{\left[\sigma - \sigma^*\left(\frac{\omega_c}{\omega_c^*}\right)\right]^2} \qquad \frac{\omega_c}{\omega_c^*} = \frac{m^*}{m}$$

$$\rho = \frac{\sigma + \sigma^*\left(\frac{m^*}{m}\right)^2}{\left[\sigma + \sigma^*\left(\frac{m^*}{m}\right)\right]^2} = \frac{m^2\sigma + m^{*2}\sigma^*}{\left[m\sigma - m^*\sigma^*\right]^2}$$

now $\sigma = \frac{ne^2\tau}{m}$, $\sigma^* = \frac{n^*e^2\tau}{m^*}$

$$\rho = \frac{mne^2\tau + m^*n^*e^2\tau}{(ne^2\tau - n^*e^2\tau)^2} = \frac{mn + m^*n^*}{(n-n^*)^2 e^2\tau}$$

$$\boxed{\rho = \frac{mn + m^*n^*}{(n-n^*)^2 e^2\tau}}$$

for large H
$\omega_c\tau \gg 1$ and $\omega_c^*\tau \gg 1$

Compare this to

$$\frac{1}{\sigma_{dc}} = \frac{1}{\sigma + \sigma^*} = \frac{1}{\frac{ne^2\tau}{m} + \frac{n^*e^2\tau}{m^*}} = \frac{1}{\left(\frac{n}{m} + \frac{n^*}{m^*}\right)e^2\tau}$$

so unlike what we found for the single carrier case, here the magnetoresistance

$$\rho \neq \frac{1}{\sigma_{dc}}$$

2) For $\vec{H} = H\hat{z}$ uniform and $\vec{E}(t) = Re\left[\vec{E}_0 e^{-i\omega t}\right]$ with $\vec{E}_0$ in the xy plane,

the Drude equation of motion is:

$$m\frac{d\vec{v}}{dt} = \vec{F}_{ext} - \frac{m\vec{v}}{\tau} = -e\vec{E} - \frac{e\vec{v}}{c}\times H\hat{z} - \frac{m\vec{v}}{\tau}$$

Assume a solution for $\vec{v}(t)$ that oscillates with the same frequency ω as the electric field $\vec{E}(t)$

$$\vec{v}(t) = Re\left[\vec{v}_0 e^{-i\omega t}\right]$$

Substitute into Drude equation to get:

$$-i\omega m\vec{v}_0 = -e\vec{E}_0 - \frac{e\vec{v}_0}{c}\times H\hat{z} - \frac{m\vec{v}_0}{\tau}$$

or

$$\vec{v}_0(1 - i\omega\tau) + \omega_c\tau\,\vec{v}_0\times\hat{z} = -\frac{e\tau}{m}\vec{E}_0$$

where $\omega_c = eH/mc$ is the cyclotron frequency. For $\vec{E}_0$ in the xy plane, the above gives $v_z = 0$. For the components in the xy plane we have

$$(1 - i\omega\tau)v_{0x} + \omega_c\tau v_{0y} = -\frac{e\tau}{m}E_{0x}$$

$$-\omega_c\tau v_{0x} + (1 - i\omega\tau)v_{0y} = -\frac{e\tau}{m}E_{0y}$$

or in matrix form

$$\begin{pmatrix} 1-i\omega\tau & \omega_c\tau \\ -\omega_c\tau & 1-i\omega\tau \end{pmatrix}\begin{pmatrix} v_{0x} \\ v_{0y} \end{pmatrix} = -\frac{e\tau}{m}\begin{pmatrix} E_{0x} \\ E_{0y} \end{pmatrix}$$

The electric current is given by $\vec{j} = -en\vec{v}$ with n the electron density. For $\vec{j}(t) = \mathrm{Re}\left[\vec{j}_0 e^{-i\omega t}\right]$ the above gives

$$\begin{pmatrix} 1-i\omega\tau & \omega_c\tau \\ -\omega_c\tau & 1-i\omega\tau \end{pmatrix}\begin{pmatrix} j_{0x} \\ j_{0y} \end{pmatrix} = \frac{ne^2\tau}{m}\begin{pmatrix} E_{0x} \\ E_{0y} \end{pmatrix}$$

We can invert the matrix to get

$$\begin{pmatrix} j_{0x} \\ j_{0y} \end{pmatrix} = \frac{\sigma_{dc}}{(1-i\omega\tau)^2 + (\omega_c\tau)^2}\begin{pmatrix} 1-i\omega\tau & -\omega_c\tau \\ \omega_c\tau & 1-i\omega\tau \end{pmatrix}\begin{pmatrix} E_{0x} \\ E_{0y} \end{pmatrix}$$

where $\sigma_{dc} \equiv ne^2\tau/m$ is the dc conductivity when $H=0$.

The above thus gives the conductivity tensor

$$\vec{j} = \overleftrightarrow{\sigma}(\omega)\cdot\vec{E}$$

$$\boxed{\overleftrightarrow{\sigma}(\omega) = \frac{\sigma_{dc}}{(1-i\omega\tau)^2 + (\omega_c\tau)^2}\begin{pmatrix} 1-i\omega\tau & -\omega_c\tau \\ \omega_c\tau & 1-i\omega\tau \end{pmatrix}}$$

b) For a transverse EM wave propagating with $\vec{k} = k\hat{z}$ along the direction of the uniform static magnetic field $\vec{H} = H\hat{z}$, we make the same assumptions as in lecture:

(i) The force on the electrons from the magnetic field part of the EM wave can be ignored

(ii) For $\lambda \gg a_0$ (Bohr radius) we can ignore the spatial variation of the electric field of the EM wave when considering its effect on the motion of an individual electron.

For a wave with fields $\vec{E}(\vec{r},t) = \mathrm{Re}\left[\vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$
and $\vec{H}(\vec{r},t) = \mathrm{Re}\left[\vec{H}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$ ← field of EM wave not unifm $H\hat{z}$

The current flowing will then be

$$\vec{j}(\vec{r},t) = \mathrm{Re}\left[\vec{j}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$

where $\vec{j}_0 = \overleftrightarrow{\sigma}(\omega)\cdot\vec{E}_0$

and $\overleftrightarrow{\sigma}(\omega)$ is the conductivity tensor of part (a)

We now proceed as in lecture

Faraday's Law: $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{H}}{\partial t} \Rightarrow i\vec{k}\times\vec{E}_0 = \frac{i\omega}{c}\vec{H}_0$

Ampere's Law: $\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \Rightarrow i\vec{k}\times\vec{H}_0 = \frac{4\pi}{c}\vec{j}_0 - \frac{i\omega}{c}\vec{E}_0$

$$\Rightarrow \quad \vec{k}\times(\vec{k}\times\vec{E}_0) = \frac{\omega}{c}(\vec{k}\times\vec{H}_0) = \frac{\omega}{c}\left[-\frac{4\pi i}{c}\vec{j}_0 - \frac{\omega}{c}\vec{E}_0\right]$$

$\updownarrow$ since $\vec{k}\perp\vec{E}_0$

or

$$-k^2\vec{E}_0 = -\frac{\omega^2}{c^2}\vec{E}_0 - \frac{4\pi i\omega}{c^2}\vec{j}_0$$

$$\left(k^2 - \frac{\omega^2}{c^2}\right)\vec{E}_0 = \frac{4\pi i\omega}{c^2}\vec{j}_0$$

$$= \frac{4\pi i\omega}{c^2}\overleftrightarrow{\sigma}(\omega)\cdot\vec{E}_0$$

conductivity tensor

or

$$\left[\left(k^2 - \frac{\omega^2}{c^2}\right)\overleftrightarrow{I} - \frac{4\pi i\omega}{c^2}\overleftrightarrow{\sigma}\right]\cdot\vec{E}_0 = 0$$

identity tensor

c) For the EM wave to satisfy Maxwell's equations, the amplitude $\vec{E}_0$ must satisfy the above linear matrix equation. The only non trivial solutions (ie only solutions with $\vec{E}_0 \neq 0$) correspond to $\vec{E}_0$ in the direction of the eigenvectors of the matrix, and the eigenvalues of the matrix vanishing!

The condition for the eigenvalues to vanish will give two dispersion relations $k(\omega)$ - one for each of the two eigenvalues of the 2x2 matrix.
The directions of the corresponding eigenvectors determine the ~~orien~~ polarization of the two transverse modes.

Note $\overleftrightarrow{\sigma}$ is an antisymmetric tensor of the form

$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{pmatrix}$$

diagonal elements are equal

Then the matrix

$$\left[\left(k^2 - \frac{\omega^2}{c^2}\right) \overleftrightarrow{I} - \frac{4\pi i \omega}{c^2} \overleftrightarrow{\sigma} \right]$$

has the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

where

$$a = k^2 - \frac{\omega^2}{c^2} - \frac{4\pi i \omega}{c^2} \sigma_{xx}$$

$$b = -\frac{4\pi i \omega}{c^2} \sigma_{xy}$$

One can check that the eigenvectors of any such matrix are proportional to $\begin{pmatrix} 1 \\ \pm i \end{pmatrix}$ and the corresponding eigenvalues are $(a \pm i b)$

- The transverse modes $\vec{E}_0 \propto \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$ correspond to left and right handed circularly polarized waves (y-component of field is 90° out of phase with x-component of field) the dispersion relations $k(\omega)$ are determined by setting the eigenvalues to zero

$$a \pm i b = 0$$

$$\Rightarrow \; k^2 - \frac{\omega^2}{c^2} - \frac{4\pi i \omega}{c^2} \sigma_{xx} \pm \frac{4\pi\omega}{c^2} \sigma_{xy} = 0$$

or

$$k_\pm^2(\omega) = \frac{\omega^2}{c^2}\left[1 + \frac{4\pi i}{\omega}\left(\sigma_{xx} \pm i \sigma_{xy}\right)\right]$$

The right hand side is a function of ω but not k. Since $\overleftrightarrow{\sigma}$ depends on ω only.

The above thus gives the two dispersion relations for left and right handed circularly polarized waves respectively

$$k_\pm(\omega) = \frac{\omega}{c}\left[1 + \frac{4\pi i}{\omega}\left(\sigma_{xx}(\omega) \pm i\sigma_{xy}(\omega)\right)\right]^{1/2}$$

The phase velocity v_p is

$$v_{p\pm} = \frac{\omega}{k_\pm} = \frac{c}{\left[1 + \frac{4\pi i}{\omega}\left(\sigma_{xx} \pm i \sigma_{xy}\right)\right]^{1/2}}$$

is different for the two modes, so left and right handed circularly polarized waves will travel with different speeds through the metal, in the presence of a uniform applied magnetic field $H\hat{z}$.

d) We have

$$c^2k^2 = \omega^2 + 4\pi i\omega\,(\sigma_{xx} \pm i\sigma_{xy})$$

with
$$\sigma_{xx} = \frac{\sigma_{dc}\,(1-i\omega\tau)}{(1-i\omega\tau)^2 + (\omega_c\tau)^2}$$

$$\sigma_{xy} = \frac{\sigma_{dc}\,(-\omega_c\tau)}{(1-i\omega\tau)^2 + (\omega_c\tau)^2}$$

So
$$\sigma_{xx} \pm i\sigma_{xy} = \frac{\sigma_{dc}\,(1 - i\omega\tau \mp i\omega_c\tau)}{(1-i\omega\tau)^2 + (\omega_c\tau)^2}$$

For large H so that $\omega_c\tau \gg 1$

and large ω so that $\omega \gg \omega_c$

$$\sigma_{xx} \pm i\sigma_{xy} \approx -\frac{\sigma_{dc}\, i\tau\,(\omega \pm \omega_c)}{(\omega_c\tau)^2 - (\omega\tau)^2}$$

$$= -\frac{i\,\sigma_{dc}\,\tau\,(\omega \pm \omega_c)}{\tau^2\,(\omega_c+\omega)(\omega_c-\omega)}$$

$$= \frac{i\,\sigma_{dc}}{\tau}\,\frac{(\omega \pm \omega_c)}{(\omega+\omega_c)(\omega-\omega_c)}$$

$$= \frac{i\,\sigma_{dc}}{\tau}\,\frac{1}{(\omega \mp \omega_c)}$$

$$c^2k^2 = \omega^2 - \frac{4\pi\omega\sigma_{dc}}{\tau}\frac{1}{(\omega \mp \omega_c)}$$

$$4\pi\frac{\sigma_{dc}}{\tau} = \frac{4\pi n e^2 \tau}{m}\frac{1}{\tau} = \frac{4\pi n e^2}{m} = \omega_p^2$$

$$c^2k^2 = \omega^2 - \omega_p^2 \frac{1}{\left(1 \mp \frac{\omega_c}{\omega}\right)}$$

For $\omega \gg \omega_c$ $\quad \frac{1}{1 \mp \frac{\omega_c}{\omega}} \simeq 1 \pm \frac{\omega_c}{\omega}$

$$c^2k^2 = \omega^2 - \omega_p^2\left(1 \pm \frac{\omega_c}{\omega}\right)$$

$$\boxed{c^2k^2 = \omega^2 - \omega_p^2 \mp \frac{\omega_c\omega_p^2}{\omega}}$$

large H $\quad \omega_c\tau \gg 1$
large ω $\quad \omega \gg \omega_c$

Now consider the opposite limit of small $\omega \ll \omega_c$
but still $\omega_c\tau \gg 1$

$$\sigma_{xx} + i\sigma_{xy} \simeq \frac{\sigma_{dc}(1 \mp i\omega_c\tau)}{1 + (\omega_c\tau)^2}$$

$$= \frac{\sigma_{dc}(1 \mp i\omega_c\tau)}{(1 + i\omega_c\tau)(1 - i\omega_c\tau)}$$

$$= \frac{\sigma_{dc}}{1 \pm i\omega_c\tau} \simeq \frac{\pm\sigma_{dc}}{i\omega_c\tau}$$

So

$$c^2k^2 = \omega^2 \pm \frac{4\pi i\omega\,\sigma_{dc}}{i\,\omega_c\tau}$$

using $\frac{4\pi\sigma_{dc}}{\tau} = \omega_p^2$

$$c^2k^2 = \omega^2 \pm \frac{\omega\,\omega_p^2}{\omega_c}$$

for small $\omega \to 0$ the first term can be ignored compared to the second, so

$$\boxed{c^2k^2 = \pm\frac{\omega\,\omega_p^2}{\omega_c}}$$

large H $\omega_c\tau \gg 1$
small ω $\omega \ll \omega_c$

in this case $\omega \sim k^2$ rather than $\omega \sim k$

3) The 2D fermi gas

a) $n_{2D} = 2 \cdot \int\limits_{|\vec{k}|\leq k_F} \frac{dk_x}{2\pi} \int \frac{dk_y}{2\pi} = 2 \cdot \int_0^{2\pi} d\varphi \int_0^{k_F} \frac{dk\, k}{(2\pi)^2}$

$\uparrow$ for 2 spin states

$$= \frac{2 \cdot 2\pi}{(2\pi)^2} \frac{k_F^2}{2} = \boxed{\frac{k_F^2}{2\pi} = n_{2D} \qquad k_F = \sqrt{2\pi n_{2D}}}$$

also $\varepsilon_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\pi \hbar^2 n_{2D}}{m}$

b) $g(\varepsilon)d\varepsilon = 2 \cdot \int\limits_{\varepsilon \leq |\vec{k}| \leq \varepsilon + d\varepsilon} \frac{dk_x}{2\pi} \int \frac{dk_y}{2\pi} = \frac{1}{\pi} \int_k^{k+dk} dk'\, k'$ where $k = \sqrt{\frac{2m\varepsilon}{\hbar^2}}$

$dk = \frac{1}{2} \frac{1}{\frac{\sqrt{2m\varepsilon}}{\hbar}} \frac{2m}{\hbar^2} d\varepsilon$

$= \frac{1}{2} \frac{2m}{\hbar^2 k} d\varepsilon$

$$g(\varepsilon)\, d\varepsilon = \frac{1}{\pi} k\, dk$$

$$g(\varepsilon) = \frac{1}{\pi} k \frac{dk}{d\varepsilon} = \frac{1}{\pi} k \frac{1}{2} \frac{2m}{\hbar^2} \frac{1}{k}$$

$$\boxed{g(\varepsilon) = \frac{m}{\pi \hbar^2} \quad \text{a constant!}}$$

c) $\frac{E}{A} = \int_0^{\varepsilon_F} d\varepsilon\, g(\varepsilon)\, \varepsilon = \frac{m}{\pi \hbar^2} \frac{\varepsilon_F^2}{2}$

A: area of system

$$\frac{E}{A} = \frac{m}{2\pi\hbar^2}\left(\frac{\pi\hbar^2 n_{2D}}{m}\right)^2 = \frac{n_{2D}}{2}\mathcal{E}_F = \frac{\pi\hbar^2}{2m} n_{2D}^2$$

$$\boxed{\frac{\mathcal{E}}{A} = \frac{n_{2D}}{2}\mathcal{E}_F = \frac{\pi\hbar^2 n_{2D}^2}{2m}}$$

d) $p = -\left(\frac{\partial E}{\partial A}\right)_{T,N}$ ← const temperature and const # particles N

$$E = \frac{\pi\hbar^2 n_{2D}^2}{2m} A = \frac{\pi\hbar^2}{2m}\left(\frac{N}{A}\right)^2 A = \frac{\pi\hbar^2 N^2}{2m}\frac{1}{A}$$

$$\boxed{P = -\left(\frac{\partial E}{\partial A}\right)_{T,N} = \frac{\pi\hbar^2 N^2}{2mA^2} = \frac{\pi\hbar^2 n_{2D}^2}{2m} = \frac{\mathcal{E}}{A}}$$

e) $B \equiv -A\left(\frac{\partial p}{\partial A}\right)_{T,N} = -A\frac{\partial}{\partial A}\left(\frac{\pi\hbar^2 N^2}{2m A^2}\right)$

$$= \frac{2\pi\hbar^2 N^2}{2mA^2} = \boxed{\frac{\pi\hbar^2 n_{2D}^2}{m} = B = 2p = 2\frac{\mathcal{E}}{A}}$$

f) $R_{sq} = \frac{\rho}{d}$ where $\rho = \frac{m}{ne^2\tau}$ and $n = \frac{n_{2D}}{d}$

So $R_{sq} = \frac{m}{ne^2\tau d}$

Suppos $\tau = \frac{d}{v_F} = \frac{dm}{p_F} = \frac{dM}{\hbar k_F}$

Thus $R_{sq} = \frac{m}{ne^2 d} \frac{\hbar k_F}{d m} = \frac{\hbar k_F}{ne^2 d^2}$

use $k_F = \sqrt{2\pi n_{2D}} = \sqrt{2\pi n d}$ from part (a)

$$R_{sq} = \frac{\hbar \sqrt{2\pi n d}}{n e^2 d^2} = \frac{\hbar}{e^2} \frac{1}{d} \sqrt{\frac{2\pi}{n d}}$$

For a film one atom thick, $d \simeq a_0$ the atomic size. Assume that the spacing between atoms in the plane is also $\sim a_0$. Then $n \approx 1/a_0^3 = 1/d^3$

$$R_{sq} = \frac{\hbar}{e^2} \frac{1}{d} \sqrt{\frac{2\pi d^3}{d}} = \frac{\hbar}{e^2} \sqrt{2\pi}$$

$$\boxed{R_{sq} \approx \sqrt{2\pi}\, \frac{\hbar}{e^2}}$$