

## PHY 521 Solutions Set 2

- 1) For a two dimensional electron gas with density  $n$  electrons/area, in zero magnetic field  $H=0$ , we have from set 1 that the energy per area is

$$u_0 = \frac{\pi \hbar^2 n^2}{2m}$$

Now turn on  $H > 0$ . The Landau level energies are:

$$\hbar \omega_c (n + 1/2) \quad \text{where } n = 0, 1, 2, \dots$$

$$\omega_c = \frac{eH}{mc}$$

Each Landau level can hold

$$\frac{H}{\Phi_0} \text{ electrons per area where } \Phi_0 = \frac{hc}{2e}$$

If the lowest  $p$  Landau levels are completely filled and the  $(p+1)^{\text{st}}$  is a fraction  $\lambda$  filled, then the energy per area is

$$u = \frac{H}{\Phi_0} \hbar \omega_c \left[ (0+1/2) + (1+1/2) + (2+1/2) + \dots + (p-1+1/2) + \lambda(p+1/2) \right]$$

The term in the bracket is

$$\underbrace{0+1+2+\dots+p-1}_{\text{sum of first } p-1 \text{ integers}} + p\left(\frac{1}{2}\right) + \lambda(p+1/2)$$

$$= (p-1)\frac{p}{2} + p\left(\frac{1}{2}\right) + \lambda(p+1/2)$$

$$u = \frac{H}{\Phi_0} \hbar \omega_c \left[ \frac{p^2}{2} + \lambda(p + 1/2) \right]$$

Now the density of electrons is given by

$$m = \frac{H}{\Phi_0} (p + \lambda)$$

$$\text{i.e. } p = \left[ \frac{m}{(H/\Phi_0)} \right]$$

bracket means the largest integer not greater than  $m/(H/\Phi_0)$  = number filled Landau levels

$$\text{and } \lambda = \frac{m}{(H/\Phi_0)} - \left[ \frac{m}{(H/\Phi_0)} \right] = \text{fraction last Landau level that is filled}$$

We can write  $\frac{\Phi_0}{H} m - \lambda = p$  and substitute into  $u$  to get

$$u(H) = \frac{H}{\Phi_0} \hbar \omega_c \left[ \frac{1}{2} \left( \frac{\Phi_0}{H} m - \lambda \right)^2 + \lambda \left( \frac{\Phi_0}{H} m - \lambda \right) + \frac{\lambda}{2} \right]$$

$$= \frac{H}{\Phi_0} \hbar \omega_c \left[ \left( \frac{\Phi_0}{H} \right)^2 m^2 - 2\lambda \frac{\Phi_0}{H} m + \lambda^2 + 2\lambda \frac{\Phi_0}{H} m - 2\lambda^2 + \lambda \right]$$

$$= \frac{\Phi_0}{H} \frac{\hbar \omega_c}{2} m^2 + \frac{H}{\Phi_0} \frac{\hbar \omega_c}{2} [\lambda - \lambda^2]$$

use  $\Phi_0 = \frac{hc}{2e}$ ,  $\omega_c = \frac{eH}{mc}$ , to write

$$u(H) = \frac{hc}{2eH} \frac{\hbar e H}{2mc} m^2 + H \frac{2e}{hc} \frac{\hbar e H}{2mc} \lambda(1-\lambda)$$

$$\hbar = 2\pi\hbar$$

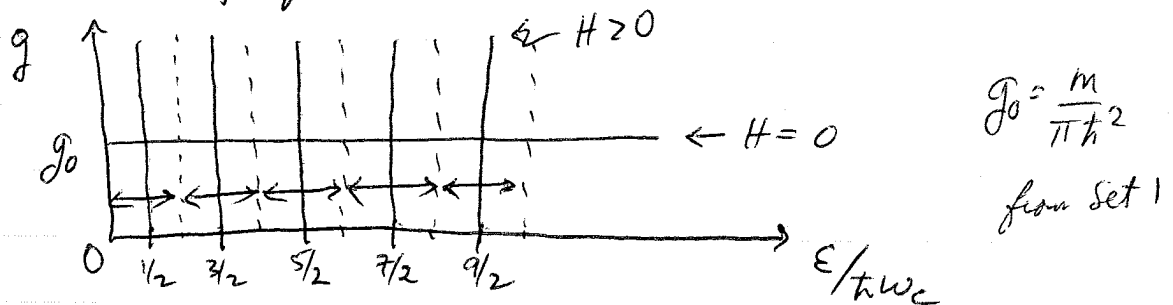
$$= \frac{\pi\hbar^2}{2m} m^2 + \frac{e^2 H^2}{2\pi mc^2} \lambda(1-\lambda)$$

the first term is just  $u_0$ , the energy density when  $H=0$   
 So

$$\Delta u(H) = u(H) - u_0 = \frac{e^2 H^2}{2\pi mc^2} \lambda(1-\lambda)$$

So  $\Delta u = 0$  whenever  $\lambda=0$  or  $\lambda=1$ , i.e. when there are only completely filled Landau levels, i.e. when  $m = p \frac{H}{\Phi_0}$  for some integer  $p$

We can understand why this must be so by comparing the density of states for  $H=0$  with  $H>0$



~~because when  $H=0$  the density of states is a constant  $g_0$ , then the energy of a filled Landau level at energy  $\hbar\omega_c(n+1/2)$  is equal in energy to the sum of energy of all  $H=0$  electron states from  $\hbar\omega_c n$  to~~

$p$  integer

Consider all the  $H=0$  electron states between  $\hbar\omega_c p$  and  $\hbar\omega_c(p+1)$ . The number of such states is

$$g_0 \hbar\omega_c = \left(\frac{m}{\pi\hbar^2}\right) \left(\hbar \frac{eH}{mc}\right) = \frac{2e}{\hbar c} H = \frac{H}{\Phi_0}$$

= density of states in one Landau level

Hence, when  $H$  is turned on, all these states bunch up into the Landau level at  $\hbar\omega_c(p+1/2)$ , and since the energy of this Landau level is in the exact middle of the energy interval  $[\hbar\omega_c p, \hbar\omega_c(p+1)]$  the total energy of these states does not change.

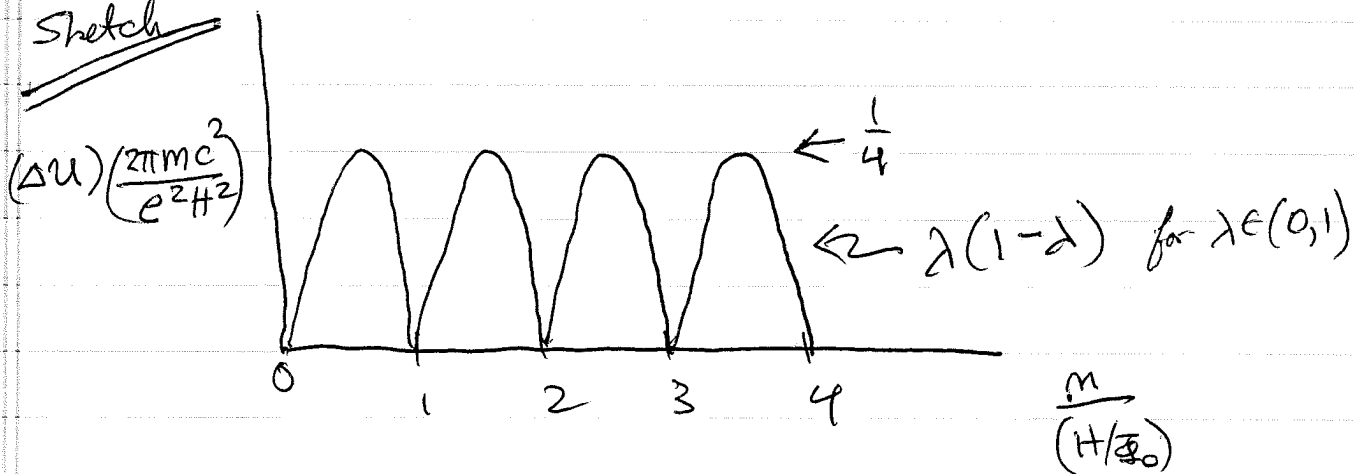
Hence  $\Delta U = 0$  whenever  $E_F^0 = \hbar\omega_c p$ ,  $p$  integer

Now  $n = \int_0^{E_F^0} dE g_0 = g_0 E_F^0$  so  $E_F^0 = \frac{n}{g_0} = \frac{\pi\hbar^2 m}{m}$

$$E_F^0 = \frac{\pi\hbar^2 m}{m} = \hbar\omega_c p = \frac{\hbar e H}{mc} p$$

$$\Rightarrow n = \frac{eH}{\pi\hbar c} p = \frac{2e}{\hbar c} H p = \left(\frac{H}{\Phi_0}\right) p$$

Sketch



$$2) \quad j_y = -\frac{e}{m} \frac{1}{LW} \sum_k \langle \psi_{nk} | \frac{\hbar}{c} \frac{\partial}{\partial y} | \psi_{nk} \rangle$$

$$j_x = -\frac{e}{m} \frac{1}{LW} \sum_k \langle \psi_{nk} | \frac{\hbar}{c} \frac{\partial}{\partial x} - \frac{e}{c} H_y | \psi_{nk} \rangle$$

with  $\psi_{nk} = C e^{ikx} \phi_n(y-y_0)$ ,  $y_0 = \frac{c}{eH} \left( \hbar k - \frac{mcE}{\hbar} \right)$

since the  $\phi_n$  are normalized, i.e.  $\int_{-\infty}^{\infty} dy \phi_n^2(y) = 1$ , then want  $\int_{-\infty}^{\infty} dx C^2 = 1$ , so  $C = \frac{1}{\sqrt{L}}$

Note that the harmonic oscillator wavefunctions are all in a definite state of parity, i.e.

$$\phi_n(y) = \pm \phi_n(-y) \quad \begin{cases} + & \text{for } n \text{ even} \\ - & \text{for } n \text{ odd} \end{cases}$$

$$\Rightarrow \frac{\partial \phi_n(y)}{\partial y} = \mp \frac{\partial \phi_n(-y)}{\partial y} \quad \begin{cases} - & \text{for } n \text{ even} \\ + & \text{for } n \text{ odd} \end{cases}$$

Consider the expectation of  $\left( \frac{\hbar}{c} \frac{\partial}{\partial y} \right)$  in state  $\psi_{nk}$

$$\langle \psi_{nk} | \frac{\hbar}{c} \frac{\partial}{\partial y} | \psi_{nk} \rangle = \int dx \int dy \psi_{nk}^* \frac{\hbar}{c} \frac{\partial}{\partial y} \psi_{nk}$$

$$= \frac{\hbar}{c} \int dx C^2 \int dy \phi_n(y-y_0) \frac{\partial \phi_n(y-y_0)}{\partial y}$$

always antisymmetric about  $y=y_0$  by parity of  $\phi_n$

$\Rightarrow$  integral vanishes for each  $\psi_{nk}$

$$\Rightarrow \boxed{j_y = 0}$$

Next consider the expectation of  $(\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - \frac{e}{c} H y)$  in  $\Psi_{nk}$

$$\begin{aligned} \langle \Psi_{nk} | \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - \frac{e}{c} H y | \Psi_{nk} \rangle &= \int dx \int dy \Psi_{nk}^* \left( \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - \frac{e}{c} H y \right) \Psi_{nk} \\ &= \underbrace{\int dx C^2}_{=1} \int dy (\hbar k - \frac{e}{c} H y) \phi_{nk}(y-y_0) \phi_{nk}(y-y_0) \end{aligned}$$

$$= \hbar k - \frac{e}{c} H \int dy y \phi_{nk}(y-y_0) \phi_{nk}(y-y_0)$$

substitute  $u = y - y_0$

$$= \hbar k - \frac{eH}{c} \int du (u + y_0) \phi_{nk}(u) \phi_{nk}(u)$$

Now  $\int du u \phi_{nk}(u) \phi_{nk}(u) = 0$  since integrand is antisymmetric about  $u = 0$

And  $\int du y_0 \phi_{nk}(u) \phi_{nk}(u) = y_0$  since  $\phi_n$  are normalized

So

$$\langle \Psi_{nk} | \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - \frac{e}{c} H y | \Psi_{nk} \rangle = \hbar k - \frac{eH}{c} y_0$$

$$= \hbar k - \frac{eH}{c} \frac{c}{eH} \left( \hbar k - \frac{mcE}{H} \right)$$

$$= \frac{mcE}{H} \quad \text{for all } \Psi_{nk}$$

$$\Rightarrow f_x = -\frac{e}{m} \frac{1}{LW} \sum_k \frac{mcE}{H} = -\frac{ecE}{H} \frac{1}{LW} \sum_k 1$$

but  $\sum_k 1 =$  number of states in Landau level

$$\sum_k 1 = (LW) \left( \frac{H}{\Phi_0} \right) \text{ so finally}$$

$$j_x = \frac{-ecE}{H} \frac{H}{\Phi_0} = \frac{-ecE}{\Phi_0} = \frac{-ecE}{hc/e}$$

$$\boxed{j_x = -\frac{e^2 E}{h}}$$

Now  $\vec{j} \cdot \vec{j} = \vec{E}$ . For  $\vec{E} = \begin{pmatrix} 0 \\ E \end{pmatrix}$  in  $\hat{y}$  direction we have

$$\begin{pmatrix} f_{xx} & f_{xy} \\ -f_{xy} & f_{yy} \end{pmatrix} \begin{pmatrix} j_x \\ j_y \end{pmatrix} = \begin{pmatrix} 0 \\ E \end{pmatrix}$$

But from above,  $j_y = 0$ , so above is

$$\begin{pmatrix} f_{xx} & f_{xy} \\ -f_{xy} & f_{yy} \end{pmatrix} \begin{pmatrix} j_x \\ 0 \end{pmatrix} = \begin{pmatrix} f_{xx} j_x \\ -f_{xy} j_x \end{pmatrix} = \begin{pmatrix} 0 \\ E \end{pmatrix}$$

$$\Rightarrow \boxed{f_{xx} = 0} \text{ since } j_x \neq 0$$

$$f_{xy} = \frac{-E}{j_x} = \frac{-E h}{-e^2 E} = \boxed{\frac{h}{e^2} = f_{xy}}$$

↑  
for a single filled  
Landau level

If there are  $s$  filled Landau levels then  $\hat{j}_y = 0$  and  $\hat{j}_x = -\frac{e^2 E}{h} s$  since each Landau level contributes to the current equally.

$$\Rightarrow \boxed{\rho_{xx} = 0, \quad \rho_{xy} = \frac{-E}{j_x} = \frac{h}{se^2}}$$

The above calculation really only works when the Landau levels are completely filled. In a real system there will always be impurities that cause scattering (and hence in general electrical resistance). The effect of this scattering has been ignored in the above calculation. But if all Landau levels are completely filled, with a finite energy gap  $\hbar\omega_c$  to the first unoccupied electron state, then none of the electrons will be able to scatter because there are no unoccupied states for them to scatter into (assuming  $E$  is not too large). Hence we can ignore the effects of scattering and our calculation above will hold.



3) a) Induced surface charge in the metallic film at  $z=0$  is

$$\delta\rho(\vec{r}_\perp) \delta(z) \quad \vec{r}_\perp = (x, y), \quad \delta(z) \text{ is Dirac delta}$$

to confine charge to  $z=0$   
gives rise to induced potential  $\delta\phi(\vec{r}_\perp, z)$  determined by

$$-\nabla^2 \delta\phi(\vec{r}_\perp, z) = 4\pi \delta\rho(\vec{r}_\perp) \delta(z)$$

Define Fourier transforms

$$\delta\phi(\vec{r}_\perp, z) = \int \frac{d^2 k_\perp}{(2\pi)^2} \int \frac{dk_z}{2\pi} e^{i\vec{k}_\perp \cdot \vec{r}_\perp} e^{ik_z z} \delta\phi(\vec{k}_\perp, k_z)$$

$$\delta\rho(\vec{r}_\perp) \delta(z) = \int \frac{d^2 k_\perp}{(2\pi)^2} \int \frac{dk_z}{2\pi} e^{i\vec{k}_\perp \cdot \vec{r}_\perp} e^{ik_z z} \delta\rho(\vec{k}_\perp)$$

Plug into Poisson's Equation to get

$$(k_\perp^2 + k_z^2) \delta\phi(\vec{k}_\perp, k_z) = 4\pi \delta\rho(\vec{k}_\perp)$$

$$\delta\phi(\vec{k}_\perp, k_z) = \frac{4\pi \delta\rho(\vec{k}_\perp)}{k_\perp^2 + k_z^2}$$

Transform back from  $k_z$  to real space  $z$  coordinate

$$\begin{aligned} \delta\phi(\vec{k}_\perp, z) &= \int \frac{dk_z}{2\pi} e^{ik_z z} \delta\phi(\vec{k}_\perp, k_z) \\ &= \int \frac{dk_z}{2\pi} e^{ik_z z} \frac{4\pi \delta\rho(\vec{k}_\perp)}{k_\perp^2 + k_z^2} \end{aligned}$$

$$\delta\phi(\vec{k}_\perp, z) = 4\pi \delta\rho(\vec{k}_\perp) \int \frac{dk_z}{2\pi} \frac{e^{ik_z z}}{k_\perp^2 + k_z^2}$$

In the plane of the film  $z=0$  we have

$$\begin{aligned} \delta\phi(\vec{k}_\perp, z=0) &= 4\pi \delta\rho(\vec{k}_\perp) \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \frac{1}{(k_z + ik_\perp)(k_z - ik_\perp)} \\ &= \frac{4\pi \delta\rho(\vec{k}_\perp)}{2k_\perp} = \frac{2\pi \delta\rho(\vec{k}_\perp)}{k_\perp} \end{aligned} \quad k_\perp = |\vec{k}_\perp|$$

So

$$\delta\phi(\vec{k}_\perp, z=0) = \frac{2\pi \delta\rho(\vec{k}_\perp)}{k_\perp}$$

b) Define the susceptibility

$$\chi(\vec{k}_\perp) = -\frac{\delta\rho(\vec{k}_\perp)}{\phi_{\text{tot}}(\vec{k}_\perp)}$$

where  $\phi_{\text{tot}}(\vec{k}_\perp) = \phi_{\text{ext}}(\vec{k}_\perp) + \delta\phi(\vec{k}_\perp)$  ← all evaluated at  $z=0$

then by (a)

$$\phi_{\text{tot}} = \phi_{\text{ext}} + \frac{2\pi \delta\rho}{k_\perp} = \phi_{\text{ext}} - \frac{2\pi \chi \phi_{\text{tot}}}{k_\perp}$$

$$\text{So } \phi_{\text{tot}} \left(1 + \frac{2\pi \chi}{k_\perp}\right) = \phi_{\text{ext}}$$

$$\Rightarrow \phi_{\text{tot}} = \frac{\phi_{\text{ext}}}{\epsilon}$$

$$\text{where } \epsilon(\vec{k}_\perp) = 1 + \frac{2\pi \chi(\vec{k}_\perp)}{k_\perp}$$

c)  $\chi(\vec{k}_\perp)$  is determined from perturbation theory exactly as we did in 3D when deriving the Lindhard dielectric function. There is no difference in 2D, except that the sum is over  $\vec{q} = (q_x, q_y)$

$$\chi(\vec{k}_\perp) = \frac{2e^2}{\text{area} \rightarrow A} \sum_{\substack{\vec{q} \\ \text{spins}}} f_{\vec{q}} \left\{ \frac{1}{\epsilon_{\vec{q}+\vec{k}_\perp} - \epsilon_{\vec{q}}} + \frac{1}{\epsilon_{\vec{q}-\vec{k}_\perp} - \epsilon_{\vec{q}}} \right\}$$

$\epsilon_{\vec{q}}$  are the single electron energies

d) We can rewrite the sum above by taking  $\vec{q}-\vec{k}_\perp \rightarrow \vec{q}$  in the 2nd term, just like we did in lecture for 3D, to get

$$\chi(\vec{k}_\perp) = \frac{2e^2}{A} \sum_{\vec{q}} \left( \frac{f_{\vec{q}} - f_{\vec{q}+\vec{k}_\perp}}{\epsilon_{\vec{q}+\vec{k}_\perp} - \epsilon_{\vec{q}}} \right)$$

For small  $k_\perp$ , expand  $f_{\vec{q}+\vec{k}_\perp} \approx f_{\vec{q}} + \frac{\partial f}{\partial \epsilon} \frac{\partial \epsilon}{\partial \vec{q}} \cdot \vec{k}_\perp$

$$\epsilon_{\vec{q}+\vec{k}_\perp} \approx \epsilon_{\vec{q}} + \frac{\partial \epsilon}{\partial \vec{q}} \cdot \vec{k}_\perp$$

where we used the fact that  $f_{\vec{q}}$  depends on  $\vec{q}$  only via  $\epsilon_{\vec{q}}$ .

$$\chi(\vec{k}_\perp) = \frac{2e^2}{A} \sum_{\vec{q}} \left( \frac{-\frac{\partial f}{\partial \epsilon} \frac{\partial \epsilon}{\partial \vec{q}} \cdot \vec{k}_\perp}{\frac{\partial \epsilon}{\partial \vec{q}} \cdot \vec{k}_\perp} \right) = e^2 \sum_{\vec{q}} \left( \frac{-\partial f}{\partial \epsilon} \right)$$

$$\text{As } T \rightarrow 0, \quad -\frac{\partial f}{\partial \epsilon} \rightarrow \delta(\epsilon - \epsilon_F)$$

$$\chi(\vec{k}_\perp) = \frac{2e^2}{A} \sum_{\vec{q}} \delta(\epsilon_{\vec{q}} - \epsilon_F) = e^2 \underset{\uparrow}{g}(\epsilon_F)$$

density of states at  $\epsilon_F$

So

$$\epsilon(\bar{k}_\perp) = 1 + \frac{2\pi\chi(k_\perp^2)}{k_\perp} = 1 + \frac{2\pi e^2 g(E_F)}{k_\perp}$$

$$\epsilon(\bar{k}_\perp) = 1 + \frac{k_s}{k_\perp}$$

$$\text{with } k_s \equiv 2\pi e^2 g(E_F)$$

$r_0 \equiv \frac{1}{k_s}$  is the characteristic screening length of the screened Coulomb interaction between electrons in the thin metallic film.

For free electrons with  $\epsilon_f = \frac{\hbar^2 q^2}{2m}$ , we have

$$g(\epsilon) = g_0 = \frac{m}{\pi\hbar^2} \text{ a constant independent of } \epsilon$$

So

$$k_s = 2\pi e^2 g_0 = \frac{2\pi e^2 m}{\pi\hbar^2} = \frac{2me^2}{\hbar^2}$$

screening length

$$r_0 = \frac{1}{k_s} = \frac{\hbar^2}{2me^2} = \frac{a_0}{2}$$

where  $a_0$  is the Bohr radius  $\sim \frac{1}{2}\text{\AA}$ .

Unlike what we found in 3D, here the screening length is independent of the electron density!  $r_0 = \frac{a_0}{2}$

e) For a point charge  $Q$  at the origin in the plane of the film,

$$\Phi_{\text{ext}}(\vec{k}_{\perp}, k_z) = \frac{4\pi Q}{k_{\perp}^2 + k_z^2} \quad \text{ordinary Coulomb potential from a point charge}$$

$$\begin{aligned} \Rightarrow \Phi_{\text{ext}}(\vec{k}_{\perp}, z=0) &= \int \frac{dk_z}{2\pi} \left( \frac{4\pi Q}{k_{\perp}^2 + k_z^2} \right) \\ &= \frac{2\pi Q}{k_{\perp}} \end{aligned}$$

So

$$\Phi_{\text{tot}}(\vec{k}_{\perp}, z=0) = \frac{\Phi_{\text{ext}}(\vec{k}_{\perp}, z=0)}{\epsilon(\vec{k}_{\perp})}$$

$$= \frac{2\pi Q}{k_{\perp}} \frac{1}{(1 + k_s/k_{\perp})} = \frac{2\pi Q}{k_{\perp} + k_s}$$

in real space

$$\Phi_{\text{tot}}(\vec{r}_{\perp}, z=0) = \int \frac{d^2 k_{\perp}}{(2\pi)^2} e^{i\vec{k}_{\perp} \cdot \vec{r}_{\perp}} \frac{2\pi Q}{k_{\perp} + k_s}$$

use polar coordinates

$$= 2\pi Q \int_0^{\infty} \frac{dk_{\perp}}{2\pi} k_{\perp} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i k_{\perp} r_{\perp} \cos\theta} \frac{1}{k_{\perp} + k_s}$$

let  $x = k_{\perp} r_{\perp}$

$$= \frac{2\pi Q}{r_{\perp}} \int_0^{\infty} \frac{dx}{2\pi} x \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{e^{ix \cos\theta}}{x + k_s r_{\perp}}$$

do the angular integral  $\int_0^{2\pi} \frac{d\theta}{2\pi} e^{ix \cos\theta} = J_0(x)$

$$\Phi_{\text{tot}}(\vec{r}_\perp, z=0) = \frac{Q}{r_\perp} \int_0^\infty dx \frac{x}{x + k_s r_\perp} J_0(x)$$

where  $J_0(x)$  is a Bessel function of the 1st kind

$$\Phi_{\text{tot}}(\vec{r}_\perp, z=0) = \frac{Q}{r_\perp} \int_0^\infty dx \left[ 1 - \frac{k_s r_\perp}{x + k_s r_\perp} \right] J_0(x)$$

$$= \frac{Q}{r_\perp} \left\{ \int_0^\infty dx J_0(x) - k_s r_\perp \int_0^\infty dx \frac{J_0(x)}{x + k_s r_\perp} \right\}$$

Bessel functions are normalized, so  $\int_0^\infty dx J_0(x) = 1$

$$\Phi_{\text{tot}}(\vec{r}_\perp, z=0) = \frac{Q}{r_\perp} \left\{ 1 - k_s r_\perp \int_0^\infty dx \frac{J_0(x)}{x + k_s r_\perp} \right\}$$

One can look up this integral in an integral handbook  
for example Gradshteyn and Ryzhik (6.562)

$$\int_0^\infty \frac{x^\nu J_0(ax)}{x + \alpha} = \frac{\pi \alpha^\nu}{2 \cos(\nu\pi)} \left[ \underset{\substack{\uparrow \\ \text{Struve} \\ \text{function}}}{H_{-\nu}(a\alpha)} - \underset{\substack{\uparrow \\ \text{Neumann} \\ \text{function}}}{N_{-\nu}(a\alpha)} \right]$$

we want  $\nu = 0$ ,  $\alpha = r_\perp k_s$ ,  $a = 1$

$$\int_0^\infty \frac{dx J_0(x)}{x + r_\perp k_s} = \frac{\pi}{2} \left[ H_0(k_s r_\perp) - N_0(k_s r_\perp) \right]$$

Looking up the properties of  $H$  and  $N$  one finds  
 $x H_0(x)$  and  $x N_0(x) \rightarrow 0$  as  $x \rightarrow 0$

So for small  $k_s r_{\perp} \ll 1$  these terms vanish, and

$$\boxed{\phi_{\text{tot}}(\vec{r}_{\perp}, z=0) \approx \frac{Q}{r_{\perp}}} \quad \text{unscreened Coulomb potential}$$

For large  $x$  one finds (see Gradshteyn and Ryzhik 8.554)

$$\begin{aligned} H_0(x) &= N_0(x) + \frac{1}{\pi} \frac{\Gamma(1/2)}{\Gamma(1/2)} \left(\frac{x}{2}\right)^{-1} + \frac{1}{\pi} \frac{\Gamma(3/2)}{\Gamma(-1/2)} \left(\frac{x}{2}\right)^{-3} \\ &= N_0(x) + \frac{2}{\pi x} - \frac{1}{4\pi} \frac{8}{x^3} \end{aligned} \quad \Gamma(x) = \text{gamma function}$$

So at large  $k_s r_{\perp} \gg 1$

$$\begin{aligned} \phi_{\text{tot}}(\vec{r}_{\perp}, z=0) &= \frac{Q}{r_{\perp}} \left[ 1 - k_s r_{\perp} \frac{\pi}{2} \left( \frac{2}{\pi r_{\perp} k_s} - \frac{2}{\pi} \frac{1}{(r_{\perp} k_s)^3} \right) \right] \\ &= \frac{Q}{r_{\perp}} \left[ 1 - 1 + \frac{1}{(k_s r_{\perp})^2} \right] \end{aligned}$$

$$\boxed{\phi_{\text{tot}}(\vec{r}_{\perp}, z=0) = \frac{Q}{k_s^2 r_{\perp}^3}}$$

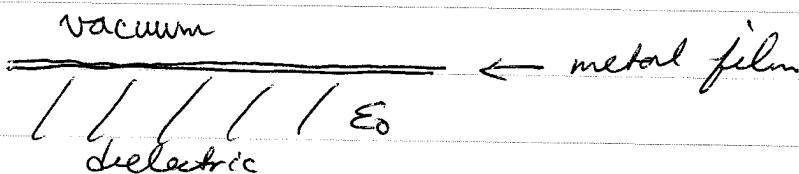
So for  $r_{\perp} \ll \frac{1}{k_s}$  the screening length, the potential between electrons in the film is the unscreened Coulomb potential  $\sim 1/r_{\perp}$ . But for  $r_{\perp} \gg \frac{1}{k_s}$ , the screened potential decays as  $\sim 1/r_{\perp}^3$ . This is faster decay than the bare Coulomb potential but still much stronger than the exponential decay we found for the screened Coulomb interaction in a bulk metal!  $1/k_s = \frac{\alpha_0}{2}$  is the distance at which one crosses over from bare to screened interaction.

f) Now suppose the metal film is deposited on top of a dielectric insulator with fixed dielectric constant  $\epsilon_0$ .

The situation is now complicated because if a surface charge  $\rho_f$  is induced in the metal film, it will polarize the bound charges in the dielectric, and so the total surface charge at the film will be

$$\rho_{f_{tot}} = \rho_f + \rho_b$$

$\uparrow$  bound charge from polarized dielectric  
 $\uparrow$  "free" charge from electrons in metal film



We need to find the relation between  $\rho_{f_{tot}}$  and  $\rho_f$ . To do this we invoke the appropriate boundary conditions at the surface charge layer.

First we find the induced potential  $\delta\phi$  from the total  $\rho_{f_{tot}}$ . The calculation is the same as in part (a).

$$-\nabla^2 \delta\phi = 4\pi \rho_{f_{tot}}(\vec{r}_\perp) \delta(z)$$

$$\delta\phi(\vec{k}_\perp, k_z) = \frac{4\pi \rho_{f_{tot}}(\vec{k}_\perp)}{k_\perp^2 + k_z^2}$$



transform  $k_z \rightarrow$  real space  $z$

$$\delta\phi(\vec{k}_\perp, z) = 4\pi \delta\rho_{\text{tot}}(\vec{k}_\perp) \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \frac{e^{ik_z z}}{(k_z - ik_\perp)(k_z + ik_\perp)}$$

do by contour integration

$$= \frac{4\pi \delta\rho_{\text{tot}}(\vec{k}_\perp)}{2\pi} 2\pi i \begin{cases} \frac{e^{-k_\perp z}}{2ik_\perp} & z > 0 \\ \frac{e^{k_\perp z}}{2ik_\perp} & z < 0 \end{cases}$$

$$\Rightarrow \delta\phi(\vec{k}_\perp, z) = \frac{2\pi \delta\rho_{\text{tot}}(\vec{k}_\perp)}{k_\perp} e^{-k_\perp |z|}$$

We can now compute the normal component of the induced electric field

$$E_z = -\frac{\partial \delta\phi}{\partial z} = \frac{2\pi \delta\rho_{\text{tot}}(\vec{k}_\perp)}{k_\perp} \times \begin{cases} k_\perp e^{-k_\perp z} & z > 0 \\ -k_\perp e^{k_\perp z} & z < 0 \end{cases}$$
$$= \pm 2\pi \delta\rho_{\text{tot}}(\vec{k}_\perp) e^{-k_\perp |z|} \begin{cases} + & \text{for } z > 0 \\ - & \text{for } z < 0 \end{cases}$$

From above we get that the jump in the normal component of  $\vec{E}$  just gives the total surface charge  $\delta\rho_{\text{tot}}$ , as expected

$$E_z(\vec{k}_\perp, z=0^+) - E_z(\vec{k}_\perp, z=0^-) = 2\pi \delta\rho_{\text{tot}}(\vec{k}_\perp) - (-2\pi \delta\rho_{\text{tot}}(\vec{k}_\perp))$$
$$= 4\pi \delta\rho_{\text{tot}}(\vec{k}_\perp)$$

The "free" charge  $\delta\rho_f$  will be given by the jump in the electric displacement  $\vec{D}$ .

$$\text{For } z > 0 \text{ in vacuum } \vec{D} = \vec{E}$$

$$\text{For } z < 0 \text{ in dielectric } \vec{D} = \epsilon_0 \vec{E}$$

The jump in the normal component of  $\vec{D}$  is then

$$D_z(\vec{k}_\perp, z=0^+) - D_z(\vec{k}_\perp, z=0^-) = 2\pi\delta\rho_{\text{tot}}(\vec{k}_\perp) - - 2\pi\delta\rho_{\text{tot}}(\vec{k}_\perp)\epsilon_0$$

$$= 2\pi\delta\rho_{\text{tot}}(\vec{k}_\perp)(1 + \epsilon_0)$$

$$= 4\pi\delta\rho_f(\vec{k}_\perp)$$

↪ the "free" surface charge due to electrons in the metal

From the above we get the desired relation between  $\delta\rho_{\text{tot}}$  and  $\delta\rho_f$

$$\delta\rho_{\text{tot}}(\vec{k}_\perp) = \frac{2\delta\rho_f(\vec{k}_\perp)}{1 + \epsilon_0}$$

{ In general  $\epsilon_0 > 1$   
so  $\delta\rho_{\text{tot}} < \delta\rho_f$   
as expected

Proceeding now as in part (b), in the plane  $z=0$

$$\phi_{\text{tot}} = \phi_{\text{ext}} + \delta\phi = \phi_{\text{ext}} + \frac{2\pi\delta\rho_{\text{tot}}}{k_\perp}$$

$$= \phi_{\text{ext}} + \frac{4\pi}{1 + \epsilon_0} \frac{\delta\rho_f}{k_\perp}$$

$$= \phi_{\text{ext}} - \frac{4\pi}{1 + \epsilon_0} \frac{1}{k_\perp} \chi(\vec{k}_\perp) \phi_{\text{tot}}$$

where  $\chi(\vec{k}_\perp) \equiv -\frac{\delta P(\vec{k}_\perp)}{\phi_{\text{tot}}(\vec{k}_\perp)} = e^2 g(\epsilon_F)$  as before

$$\Rightarrow \phi_{\text{tot}} = \frac{\phi_{\text{ext}}}{\epsilon(\vec{k}_\perp)}$$

where

$$\begin{aligned}\epsilon(\vec{k}_\perp) &= 1 + \frac{4\pi}{1+\epsilon_0} \frac{e^2 g(\epsilon_F)}{k_\perp} \\ &= 1 + \frac{k_s}{k_\perp}\end{aligned}$$

with  $k_s = \frac{4\pi}{1+\epsilon_0} e^2 g(\epsilon_F)$

So the only change is that the screening length

$$r_0 = \frac{1}{k_s} \text{ is increased by a factor } \frac{1+\epsilon_0}{2}$$