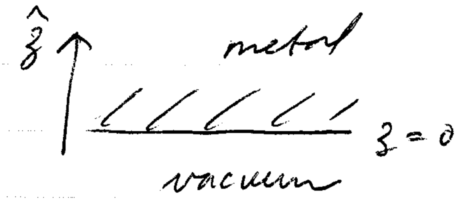
1) 

For $z>0$ write $\vec{k} = q\hat{x} + ik\hat{z}$

then $\vec{E} = (A\hat{x} + B\hat{z})\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

For $z<0$

write $\vec{k}' = q\hat{x} - ik'\hat{z}$

then $\vec{E} = (C\hat{x} + D\hat{z})\, e^{i(\vec{k}'\cdot\vec{r} - \omega t)}$

boundary conditions at $z=0$:

tangential component of $\vec{E}$ is continuous

$\Rightarrow \boxed{A = C}$ (f)

normal component of $\varepsilon\vec{E}$ is continuous

$\Rightarrow \boxed{\varepsilon B = D}$ (e)

where $\varepsilon(\omega) = 1 + \frac{4\pi i\,\sigma(\omega)}{\omega}$ is dielectric function of the metal

Our solutions must obey Maxwell's equations

$\vec{\nabla}\cdot\vec{E} = 0$ $\qquad$ $\vec{\nabla}\cdot\vec{H} = 0$

$\vec{\nabla}\times\vec{E} = \frac{-1}{c}\frac{\partial \vec{H}}{\partial t}$ $\qquad$ $\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$

For $z>0$ $\quad$ $\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \vec{k}\cdot\vec{E} = 0$

$(q\hat{x} + ik\hat{z})\cdot(A\hat{x} + B\hat{z}) = 0$

$\Rightarrow qA + ikB = 0$

$$\boxed{B = +i\frac{q}{k} A} \quad (d)$$

For $z<0$ $\quad \vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \vec{k}'\cdot\vec{E} = 0$

$$(q\hat{x} - ik'\hat{z})\cdot(C\hat{x} + D\hat{z}) = 0$$

$$\Rightarrow qC - ik'D = 0$$

use $C = A$, $D = \varepsilon B$

$$\Rightarrow qA - ik'\varepsilon B = 0$$

$$\boxed{B = -\frac{iq}{k'\varepsilon} A}$$

Comparing the last two results we get

$$\boxed{k = -k'\varepsilon} \quad (c)$$

Combine Faraday's + Ampere's laws

For $z \geq 0$ $\quad \vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\nabla^2\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{H})$

$$= -\frac{4\pi}{c^2}\frac{\partial \vec{j}}{\partial t} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\Rightarrow k^2\vec{E} = \frac{4\pi i\omega}{c^2}\sigma\vec{E} + \frac{\omega^2}{c^2}\vec{E}$$

$$\Rightarrow (q^2 - k^2)\vec{E} = \frac{\omega^2}{c^2}\left[1 + \frac{4\pi i\sigma}{\omega}\right]\vec{E}$$

$$= \frac{\omega^2}{c^2}\varepsilon(\omega)\vec{E}$$

$$\Rightarrow \boxed{q^2 - k^2 = \frac{\omega^2}{c^2}\varepsilon}$$

Similarly, <u>For $z<0$</u> we get

$$q^2 - K'^2 = \frac{\omega^2}{c^2}$$

use $K' = -\frac{K}{\varepsilon}$ $\Rightarrow$ $q^2 - \frac{K^2}{\varepsilon^2} = \frac{\omega^2}{c^2}$

$$\Rightarrow q^2\varepsilon - \frac{K^2}{\varepsilon} = \frac{\omega^2}{c^2}\varepsilon = q^2 - K^2$$

$$\Rightarrow q^2(\varepsilon - 1) = \left(\frac{1}{\varepsilon} - 1\right)K^2 = -\left(\frac{\varepsilon - 1}{\varepsilon}\right)K^2$$

$$\Rightarrow \boxed{K^2 = -\varepsilon q^2} \text{ (b)}$$

So $q^2\varepsilon - \frac{K^2}{\varepsilon} = q^2\varepsilon + q^2 = q^2(1+\varepsilon) = \frac{\omega^2}{c^2}\varepsilon$

$$\boxed{c^2 q^2 = \left(\frac{\varepsilon}{1+\varepsilon}\right)\omega^2} \text{ (a)}$$

So (a) is the dispersion relation that gives wavevector q in terms of frequency ω

(b) knowing q, we now can find the decay length $1/K$ in the metal

(c) and $1/K'$ in the metal

(d) gives the ratio of amplitudes B/A in metal

(e) and (f) gives ratios C/A and D/A in vacuum

To have a solution where q, K, K' are all real and positive,

$$K^2 = -\varepsilon q^2 \Rightarrow \text{we must have } \boxed{\varepsilon < 0}$$

so K is real

But then

$$c^2 q^2 = \left(\frac{\varepsilon}{1+\varepsilon}\right)\omega^2 \Rightarrow \text{we must have } \boxed{1+\varepsilon < 0}$$

so that q is real for real ω

For Drude model with $\omega\tau \gg 1$ we use

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$$

Condition $\varepsilon + 1 < 0 \Rightarrow 2 - \frac{\omega_p^2}{\omega^2} < 0$

$$\Rightarrow 2 < \frac{\omega_p^2}{\omega^2}$$

$$\Rightarrow \boxed{\omega < \frac{\omega_p}{\sqrt{2}}}$$

so solution is only possible for frequencies $\omega < \frac{\omega_p}{\sqrt{2}}$

$$c^2 q^2 = \left(\frac{\varepsilon}{1+\varepsilon}\right)\omega^2 = \left(\frac{1 - \frac{\omega_p^2}{\omega^2}}{2 - \frac{\omega_p^2}{\omega^2}}\right)\omega^2$$

when $\omega \to \frac{\omega_p}{\sqrt{2}}$, $1+\varepsilon \to 0$, so

$\frac{\varepsilon}{1+\varepsilon} \to \infty$ solution gives $q \to \infty$

So $\boxed{\text{as } q \to \infty, \ \omega \to \frac{\omega_p}{\sqrt{2}}}$

For small $\omega \ll \omega_p$,

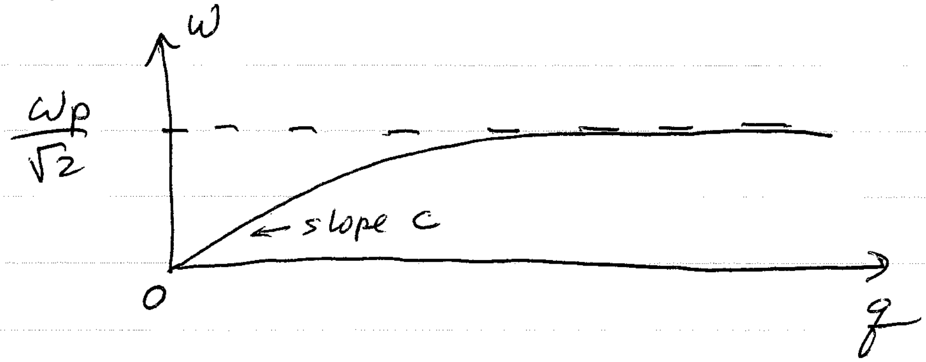
$$\frac{\varepsilon}{1+\varepsilon} = \frac{1 - \frac{\omega_p^2}{\omega^2}}{2 - \frac{\omega_p^2}{\omega^2}} \simeq \frac{\omega_p^2 - \omega^2}{\omega_p^2 - 2\omega^2} \simeq 1 \quad \text{as } \omega \to 0$$

so for small $\omega \ll \omega_p$

$$c^2 q^2 \simeq \omega^2$$

$$\boxed{cq \simeq \omega \qquad \text{small } \omega \ll \omega_p}$$

Sketch



The surface charge density on the surface of the metal is given by:

jump in normal component $E = 4\pi\sigma$

$$\Rightarrow \quad 4\pi\sigma = \hat{z} \cdot (\vec{E}_{metal} - \vec{E}_{vacuum})$$
$$= B - D$$
$$= B - \varepsilon B$$
$$= B(1-\varepsilon)$$
$$= \frac{+iq}{K}(1-\varepsilon)A$$

use $K^2 = -\varepsilon q^2$

$$4\pi\sigma = \frac{+i(1-\varepsilon)}{\sqrt{-\varepsilon}} A$$

use $\varepsilon = 1 - \left(\frac{\omega_p}{\omega}\right)^2$

"i" oscilations in σ are $\pi/2$ out of phase with oscilations in E_x

$$4\pi\sigma = \frac{i\left(\frac{\omega_p}{\omega}\right)^2}{\sqrt{\left(\frac{\omega_p}{\omega}\right)^2 - 1}} A = \boxed{\frac{i\,\omega_p^2}{\omega\sqrt{\omega_p^2 - \omega^2}} A = 4\pi\sigma}$$

small $\omega \ll \omega_p \Rightarrow 4\pi\sigma \simeq \frac{i\omega_p}{\omega} A$ grows as $\omega \to 0$

large $q \to \infty$, $\omega \to \frac{\omega_p}{\sqrt{2}} \Rightarrow 4\pi\sigma = ~~[illegible]~~ \; 2iA$

Polarization : ~~[illegible] $\frac{\omega_p}{\sqrt{2}}$~~

then $B = \frac{iq}{K} A$, $K = \sqrt{-\varepsilon}\, q$

$$\Rightarrow \quad B = \frac{i}{\sqrt{-\varepsilon}} A = \frac{i}{\sqrt{\left(\frac{\omega_p}{\omega}\right)^2 - 1}} A$$

When $q \to \infty$ $\omega \to \omega_p/\sqrt{2}$

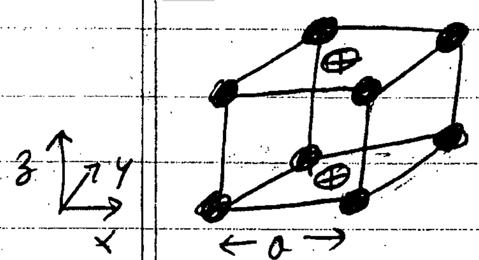
$$\boxed{B = iA}$$

$\Rightarrow$ in metal, $\vec{E} = A(\hat{x} + i\hat{z})\, e^{iqx} e^{-kz} e^{-i\omega t}$

$\rightarrow$ circularly polarized

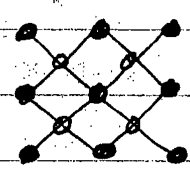
E_z oscillates $\frac{\pi}{2}$ out of phase with E_x

2) a) base centered cubic - add a site to center of all horizontal planes



$\boxed{\text{yes}}$ this is a Bravais lattice

In the xy plane the sites look like

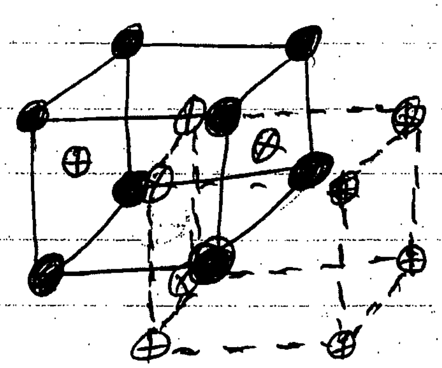


which is a square BL with lattice constant $a' = a/\sqrt{2}$

stacking such 2D square lattices in the z direction with lattice constant a thus results in a 3D BL with $\boxed{\vec{a}_1 = \frac{a}{2}(\hat{x}+\hat{y})}$, $\boxed{\vec{a}_2 = \frac{a}{2}(\hat{x}-\hat{y}),\ \vec{a}_3 = a\hat{z}}$

Note also that the ⊕ sites in 3D picture above also form a simple cubic lattice with the ● sites in the center of their horizontal faces ⇒ all sites are equivalent — another way to prove it is a B.L.

b) side centered cubic - add a site to center of all vertical planes

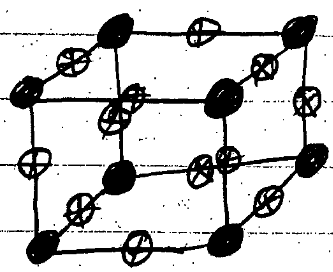


The ⊕ sides do <u>not</u> form a simple cubic lattice, rather they form the base centered cubic of part (a) ⇒ ⊕ sites do not look identical to the ● sites.

⇒ this is $\boxed{\text{NOT}}$ a Bravais lattice, but rather a lattice with a basis. We can choose

primitive vectors: $\vec{a}_1 = a\hat{x}$, $\vec{a}_2 = a\hat{y}$, $\vec{a}_3 = a\hat{z}$
3 point basis: $\vec{d}_1 = 0$, $\vec{d}_2 = \frac{a}{2}(\hat{x}+\hat{z})$, $\vec{d}_3 = \frac{a}{2}(\hat{y}+\hat{z})$

c) Edge centered cubic – add a site to middle of all edges



This is $\boxed{\text{NOT}}$ a Bravais lattice

each ● site has 6 ⊕ nearest neighbors a distance $a/2$ away. each ⊕ site has only 2 ● nearest neighbors a distance $\frac{a}{2}$ away. Hence ● sites and ⊕ sites are not equivalent

primitive vectors: $\vec{a}_1 = a\hat{x}$, $\vec{a}_2 = a\hat{y}$, $\vec{a}_3 = a\hat{z}$
4 point basis: $\vec{d}_1 = 0$, $\vec{d}_2 = \frac{a}{2}\hat{x}$, $\vec{d}_3 = \frac{a}{2}\hat{y}$, $\vec{d}_4 = \frac{a}{2}\hat{z}$