Transport Properties of the Sommerfeld Model

We have seen that we must use quantum-Fermi-Dirac statistics to describe the thermodynamic behavior of conduction electrons.

What about the equation motion? Semi-classical arguments can be made to show that in many cases it remains ok to use the classical Drude equation of motion. We need to be able to construct electron wave packets which are localized on the desired spatial length scales.

We saw that the typical electron energy is set by the Fermi energy $E_F$. Hence the typical momentum is $p \sim p_F$. If we make a wave packet with $\Delta p \ll p_F$ then spread in spatial position is

$$\Delta x \sim \frac{\hbar}{\Delta p} \gg \frac{1}{v_F}$$

So the electron cannot be localized to atomic length scales, but can be localized on the length scales of macroscopically varying electric, magnetic fields or temperature gradients $\sim 10^3 \text{ Å}$. So ok for motion in EM waves in visible spectrum.
but not for X-rays (\( \lambda_{\text{x-ray}} \approx 8 \)).

We also need \( \Delta x < l \) the mean free path. Classical motion may fail when \( l \approx 10^2 \).

\[ l = \frac{\nu t}{2} \quad \text{where} \quad \nu \sim 10^{-14} \text{ sec} \quad \text{and} \quad \frac{t}{\nu} \sim 10^8 \text{ cm/sec} \]

So \( l \sim 10^8 \cdot 10^{-14} \text{ cm} = 10^{-6} \text{ cm} = 100 \AA \)

So looks OK.

Using Drude\footnote{Drude} Equation, motion + Fermi-Dirac Statistics we then have

(1) dc or ac electric conductivity same as for classical Drude model, since no thermo was involved.

(2) For thermal conductivity we had

\[ K = \frac{1}{3} \nu^2 \tau \sigma \]

Now we should use \( \nu = \nu_F \), \( \nu_F^2 = \frac{2eF}{m} \)

and \( \sigma = \frac{\pi^2}{3} \left( \frac{k_B}{eF} \right)^2 m k_B \tau \)

\[ \Rightarrow K = \frac{1}{3} \frac{2eF}{m} \frac{\pi^2}{3} \left( \frac{k_B}{eF} \right) m k_B \tau \]

\[ = \frac{\pi^2}{3m} \tau m k_B^2 T \]
and Wiedemann–Franz coefficient is

\[ \sigma T = \frac{\pi^2}{3m} \frac{\tau m k_B^2}{m e^2 c} = \frac{\pi^2}{3} \left( \frac{k_B}{e} \right)^2 = 2.44 \times 10^{-8} \text{ ohm} \cdot \text{cm} \cdot \text{K}^{-1} \]

excellent agreement
with experiment

(3) Thermopower

\[ Q = -\sigma = -\frac{\pi^2}{3} \frac{k_B}{e} \left( \frac{k_B T}{e \phi} \right) \]

\[ = -1.42 \left( \frac{k_B T}{e \phi} \right) \times 10^{-4} \text{ volt/K} \]

more reasonable result than
classical value
Magnetic properties of Free Electron Gas

In the presence of an applied magnetic field $\vec{H}$, the electron gas will develop a net magnetization via two effects:

1) The intrinsic spins of the electrons
   - anti-align with $\vec{H} \Rightarrow$ magnetic moments
   - align with $\vec{H} \Rightarrow$ paramagnetic effect
   **Pauli Paramagnetism**

2) The electrons move in closed orbits
   $\Rightarrow$ circulating currents $\Rightarrow$ magnetic moments anti-aligned with $\vec{H} \Rightarrow$
   diamagnetic effect
   **Landau diamagnetism**

We consider first **Pauli Paramagnetism**

(CA+M Chpt 31)

An electron with intrinsic spin $\frac{1}{2}$ ($s_z = \pm 1$)
also intrinsic magnetic moment $\mu = -\mu_0 \frac{\hbar}{2mc}$

where $\mu_0 = \frac{e \hbar}{2mc}$ is the Bohr magneton

The interaction energy of the spin with the applied magnetic field is:

$$\varepsilon \vec{H} = -\mu \cdot \vec{H} = \mu_0 \frac{\hbar}{2mc} \cdot \vec{H}$$
If we take $\tilde{H} = H \hat{\sigma}$, the above $\delta \tilde{H}$ then gives the single electron energy eigenvalues

$$E_{ks} = \frac{\hbar^2 k^2}{2m} + s \mu_0 H$$

with $s = \pm 1$

(parallel) spin-up electrons, $s = +1$, increase their energy

(antiparallel) spin-down electrons, $s = -1$, decrease their energy

$\Rightarrow$ In equilibrium, this shift of electron energy with $s$ results in a net excess of $s = -1$ electron.

For $\tilde{H} = 0$, the dispersion curves look like

All states up to $E_F$ are filled

Now imagine turning on a small $\tilde{H}_1$, but keeping the $s = \pm 1$ populations of electrons the same as when $\tilde{H} = 0$.

We see that above cannot remain the ground
state, as the energy will be lowered by having up electrons at $\varepsilon + \mu H$, $S = +1$
convert into down electrons and go into the empty states at $\varepsilon - \mu H$, $S = -1$.

The ground state will instead look like

\[ \begin{align*}
    \text{down} & \quad \varepsilon & \quad \text{up} \\
    & & \leftarrow \sim \varepsilon_F
\end{align*} \]

and we thus see that there will be more down than up electrons in the ground state.

Since $S = \pm 1$, electrons have magnetic moment $\mu = -\mu_0 s$, the system has a net positive magnetization aligned with $H$.

To see how big $M$ we need to compute the number of up electrons that flip into down electrons when $H$ is turned on.

We will assume that $H$ is small enough that $\mu H \ll \varepsilon_F$. When this is so, we can ignore the fact that the density of states has a slight variation with energy $\varepsilon$ over the range $\varepsilon + \mu H$ to $\varepsilon - \mu H$ and assume it to be roughly constant $g(\varepsilon_F)$. 
The number of up electrons that flip is then easily computed from the following sketch.

\[ \text{The number of up electrons that must flip is } \]
\[ \text{therefore } \]
\[ g_+ (E_F) \Delta E \]

where \( g_+ (E_F) = \frac{1}{2} g (E_F) \) is the density of states of up electrons at \( E_F \), which is half the total density of states at \( E_F \), and \( \Delta E = \frac{1}{2} [\mu_{0} H - (-\mu_{0} H)] = \mu_{0} H \) is the energy interval that must flip. The number that flip is therefore \( \Delta N = \frac{1}{2} g (E_F) \mu_{0} H \).

In the new ground state, the number of down electrons is now \( N_0 + \Delta N \), and the number of up electrons is \( N_0 - \Delta N \), where \( N_0 = \frac{1}{2} m \) in the ground state when \( H = 0 \).
So the net magnetization is now (at $T=0$)

$$M = \mu_0 (M_+ - M_-)$$

$$= \mu_0 (m_0 + \Delta m - (m_0 - \Delta m))$$

$$= 2 \mu_0 \Delta m$$

$$= g(\varepsilon_F) \mu_0^2 H$$

and the Pauli paramagnetic susceptibility is

$$\chi_p = \frac{\partial M}{\partial H} = g(\varepsilon_F) \mu_0^2$$

proportional to the density of states at Fermi energy.

For the free electron gas we had

$$g(\varepsilon_F) = \frac{3}{2} \frac{m}{\varepsilon_F}$$

$$\chi_p = \frac{3}{2} \frac{M}{\varepsilon_F} \mu_0^2$$

$$m \sim k_F^3, \varepsilon_F \sim k_F^2,$$ so $$\chi_p \sim k_F \sim \frac{1}{(r_s/a_0)}$$

$$\chi_p = \frac{2.59 \times 10^{-6}}{(r_s/a_0)}$$

corrections to above result at finite $T$ of order

$$\left(\frac{T}{T_F}\right)^2$$ so above is very good at all $T < T_F$ and so good at room temperature.
Compare to experiment

<table>
<thead>
<tr>
<th>metal</th>
<th>$\phi/\alpha$</th>
<th>$x_p$ theory</th>
<th>$x_p$ ext</th>
<th>$x \times 10^{-6}$</th>
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<tbody>
<tr>
<td>Li</td>
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<td>0.80</td>
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<tr>
<td>Cs</td>
<td>5.62</td>
<td>0.46</td>
<td>0.8</td>
<td></td>
</tr>
</tbody>
</table>

It turns out that the discrepancy between theory and ext is mainly due to having neglected electron-electron interactions.

Note that $x_p$ above is very different from what one gets with classical statistics.

Classically

$$M \approx \left[ \frac{-\mu_0 H/k_B T + \mu_0 H/k_B T}{e^{\mu_0 H/k_B T} + e^{\mu_0 H/k_B T}} \right] (-\mu_0) m$$

$$= \left[ \frac{e^{\mu_0 H/k_B T} - e^{-\mu_0 H/k_B T}}{e^{\mu_0 H/k_B T} + e^{-\mu_0 H/k_B T}} \right] \mu_0 m$$

$$\approx \frac{2\mu_0 H M}{k_B T} \quad \text{when } \mu_0 H \ll k_B T$$

$$\frac{\chi_{\text{classical}}}{\mu_0 m} \approx \frac{\mu_0 m}{k_B T} \quad \text{Curie law } \sim \frac{1}{T}$$

So

$$\frac{x_p}{\chi_{\text{classical}}} \sim \left( \frac{T}{T_F} \right) \ll 1$$