Landau Diamagnetism - Landau Levels

Here we wish to consider the effect of the magnetic field on the orbital motion of the conduction electrons. To do so, we must solve the quantum mechanical problem of a charged particle moving in a uniform magnetic field.

The geometry we consider is:

\[
\begin{align*}
\mathbf{H} &= H \mathbf{\hat{z}} \\
\mathbf{L}_x &\rightarrow \mathbf{L}_y \\
\mathbf{L}_z &\uparrow
\end{align*}
\]

For a particle of charge \( q \) in a static uniform magnetic field, the Hamiltonian is:

\[
\hat{H} = \frac{\hbar^2}{2m} \left( \frac{\hbar}{i} \nabla - \frac{\mathbf{q}}{\mathbf{e}} \mathbf{A} \right)^2
\]

where \( \mathbf{A} \) is the vector potential, \( \mathbf{H} = \nabla \times \mathbf{A} \)

\( q = -e \) is the charge of the electron.

For \( \mathbf{H} = H \mathbf{\hat{z}} \), we will use \( \mathbf{A} = -\mathbf{y} H \mathbf{\hat{x}} \)

Substitute these into \( \hat{H} \) to get:
\[ H = \frac{1}{2m} \left( \frac{\hbar^2}{2} \frac{\partial^2}{\partial y^2} - \frac{e}{\epsilon} H x \right)^2 \]

\[ = \frac{1}{2m} \left[ -\frac{\hbar^2}{2} \frac{\partial^2}{\partial y^2} - \frac{\hbar^2}{2} \frac{\partial^2}{\partial y^2} + \left( \frac{\hbar}{2} \frac{\partial}{\partial x} - \frac{e}{\epsilon} H y \right)^2 \right] \]

We want to find the eigenstates \( \psi \) that solve

\[ H \psi = \epsilon \psi \]

\( \epsilon \) is eigenvalue of energy.

Any solution of the form

\[ \psi(x, y) = e^{ikx} e^{iky} \phi(y) \]

This form is suggested as \( H \) is translationally invariant in \( x \) and \( y \), but not in \( y \) (due to our particular choice for \( A \)).

Substituting the \( \psi \) into the above Schrodinger Equation to get

\[ \frac{1}{2m} \left[ -\frac{\hbar^2 k_x^2}{2} - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial y^2} + \left( \frac{\hbar}{2} k_x - \frac{e}{\epsilon} H y \right)^2 \right] \phi(y) = \epsilon \phi(y) \]

or

\[ -\frac{\hbar^2}{2m} \frac{\partial^2 \phi}{\partial y^2} + \frac{1}{2m} \left( \frac{\hbar}{2} k_x - \frac{e}{\epsilon} H y \right)^2 \phi = \left( \epsilon - \frac{\hbar^2 k_x^2}{2m} \right) \phi \]

Define \( y_0 \) such that \( k_x y_0 = \frac{e}{\epsilon} H y_0 \).

\[ -\frac{\hbar^2}{2m} \frac{\partial^2 \phi}{\partial y^2} + \frac{1}{2m} \left( \frac{\hbar}{2} \right)^2 \left( y - y_0 \right)^2 \phi = \left( \epsilon - \frac{\hbar^2 k_x^2}{2m} \right) \phi \]

Define cyclotron frequency \( \omega_c \equiv \frac{e H}{mc} \)

(a classical charged particle in uniform \( H \) moves in a circular orbit with angular velocity \( \omega_c \)).
Finally we get

\[ \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial y^2} + \frac{1}{2} \, m \omega_c^2 \, (y - y_0)^2 \right] \phi(y) = \left( E - \frac{\hbar^2 \omega_c^2}{2m} \right) \phi(y) \]

This is just the Hamiltonian for a single harmonic oscillator of frequency \( \omega_c \) that is centered at \( y = y_0 \).

We know the eigenvalues of energy of the harmonic oscillator are just

\[ \hbar \omega_c (n + \frac{1}{2}) \quad n = 0, 1, 2, \ldots \]

So we then have

\[ E = \frac{\hbar^2 \omega_c^2}{2m} + \hbar \omega_c \left( n + \frac{1}{2} \right) \]

or the energy eigenvalues of our particle are

\[ E = \frac{\hbar^2 \omega_c^2}{2m} + \hbar \omega_c \left( n + \frac{1}{2} \right) \]

kinetic energy \( \frac{\hbar^2 \omega_c^2}{2m} \)

potential energy \( \hbar \omega_c \left( n + \frac{1}{2} \right) \)

motion along \( y \) \( \frac{1}{2} \)

parallel to \( \mathbf{z} \)

orbital motion in \( xy \) plane \( + \) to \( \mathbf{z} \)

The wave functions \( \phi_n(y) \) are the usual harmonic oscillator wavefunctions (gaussian \( \times \) Hermite polynomial) only centered at \( y_0 \).
We can therefore write our solution in terms of the quantum numbers, $k_x, k_y, n$

\[
\psi_{k_x, k_y, n}(x, y, z) = e^{i k_x x} e^{i k_y y} \phi_n(y - y_0)
\]

($\phi$ is a 0 wave function centered at origin)

\[
E(k_x, k_y, n) = \frac{\hbar^2 k_x^2}{2m} + \hbar \omega_c (n + 1/2)
\]

where $y_0 = \hbar k_x c = \hbar k_x \frac{e^2}{\hbar}$

Note $E$ is independent of $k_x$ so for fixed $k_y$ and $n$ there are many degenerate states corresponding to the different possible choices for $k_x$.

What are the possible values of $k_x$?

If we take periodic boundary conditions along $x$, $\psi(x + l_x, y, z) = \psi(x, y, z)$ then we must have

\[
e^{i k_x x} = e^{i k_x (x + l_x)} \Rightarrow k_x = \frac{2\pi n}{l_x}
\]

But $k_x$ also determines the value of $y_0$ about which the wave function is centered in the $y$ direction. Therefore we must have

\[
0 \leq y_0 \leq L_y \Rightarrow 0 \leq \frac{\hbar k_x}{\hbar \omega_c} \leq L_y
\]

\[
\Rightarrow k_x \text{ max} = \frac{L_y \hbar \omega_c}{\hbar} = \frac{L_y \hbar e^2}{\kappa mc} = \frac{L_y e^2}{\kappa mc}
\]
Combining these two conditions we have the

number of allowed values \( k_x \) can take is given by

\[
\frac{k_{x \text{max}}}{\Delta k_x} = \frac{\frac{4\pi c}{\hbar c}}{2\pi} = \frac{\pi}{L_x} \frac{L_x}{\pi} \frac{e^H}{2\pi \hbar c} = \frac{L_x L_y e^H}{2\pi \hbar c} \]

\[
= \frac{L_x L_y H}{\frac{\hbar c}{e}}
\]

to get the number of allowed electron states with energy

\[
\frac{3}{2} k_B T + \hbar \omega_c (n + \frac{1}{2})
\]

we should multiply above by

a factor of 2 for the two possible spin states.

Degeneracy \( \mathcal{N} = 2 \frac{L_x L_y H}{\frac{\hbar c}{e}} = \frac{\Phi}{\frac{\hbar c}{2e}} = \frac{\Phi}{\Phi_0} \)

where \( \Phi = L_x L_y H \) is the total magnetic flux

penetrating the system, and

\[
\Phi_0 = \frac{\hbar c}{2e}
\]

two units of magnetic flux and is
called the flux quantum

\[
\Phi_0 = 2.07 \times 10^{-7} \text{ gauss cm}^2
\]

degeneracy \( \mathcal{N} = \frac{\Phi}{\Phi_0} = \text{number of flux quanta} \)
Consider now just the motion of the electron in the $xy$ plane. The energy of this motion is

$$\tilde{E} = E - \frac{\hbar^2 k_x^2}{2m} = \hbar \omega_c (n + \frac{1}{2}) \quad n = 0, 1, 2, \ldots$$

The states corresponding to a given value of $n$ are called the "$n$th Landau level". The $n$th Landau level has a degeneracy of $\hbar / \Phi_0$, or equivalently, the number of electrons per unit area that one can put into a given Landau level is

$$\frac{1}{\text{area}} \frac{\Phi_0}{\hbar} = \frac{H}{\Phi_0}$$

We can summarize this by giving the density of states for the energy $\tilde{E}$ in the $xy$ plane

$$g_{2D}(\tilde{E}) \Delta E = \text{number of electron states per unit area with energy in the range } \tilde{E} \text{ to } \tilde{E} + \Delta \tilde{E}.$$  

Since there are only states at the discrete energy values $\hbar \omega_c (n + \frac{1}{2})$, $g_{2D}(\tilde{E})$ is a sum of $S$-functions at these discrete values - the amplitude of each $S$-function is just the degeneracy per area $H/\Phi_0$

$$g_{2D}(\tilde{E}) = \sum_n \frac{1}{\Phi_0} S(\tilde{E} - \hbar \omega_c (n + \frac{1}{2}))$$
We can compare this to the 2D density of states when \( H = 0 \). From problem (3b) of HW set 1, you will find that at \( H = 0 \), \( g_{2D}(E) \) is constant:

\[
H = 0 : \quad g_{2D}(E) = \frac{m}{\pi \hbar^2} \quad g_{2D} \rightarrow \frac{m}{\pi \hbar^2}
\]

To compare \( H = 0 \) with \( H > 0 \), consider computing the average density of states for \( H > 0 \) where we average over an energy interval large compared to the spacing between the Landau levels. We then compute the average density of states:

\[
\bar{g} = \left( \text{# S-function spikes in } \Delta E \right) \times \frac{H}{\Phi_0} \quad \text{interval width } \Delta E
\]

If we take \( \Delta E = \hbar t \omega_c \) for a large integer \( M \), then on average there will be \( M \) S-function spikes in this interval, so

\[
\bar{g} = \frac{M}{M \hbar t \omega_c} = \frac{H}{\Phi_0} \left( \frac{e \hbar}{2e} \right) = \frac{H}{e^2} \frac{1}{\hbar^2 \omega_c} = \frac{m}{\pi \hbar^2}
\]

so average density of state at \( H > 0 \)

\[
\bar{g} = \frac{m}{\pi \hbar^2} = \text{constant density of states at } H = 0
\]
So turning on the magnetic field branches the energy eigenstates up into discrete levels, but the average number of states per unit energy remains the same (provided we average on intervals $\gg \hbar \omega$).
Suppose we had an actual 2D electron gas.

One can think of making this in a thin metallic film or a semiconductor inversion layer where the gas is confined to a region in space along \( \ell \) so small that only the lowest allowed value of \( k_z \) is occupied, i.e. \( \frac{2\pi}{\ell} = \Delta k_z \), giving larger than all other energy scales.

What is necessary so that one could detect the difference between the discrete Landau level structure at finite \( H > 0 \), and the average density of states which is equal to its \( H = 0 \) value?

If \( f \) is the Fermi function, we know that finite temperature smear out the sharp cutoff at \( E = E_F \) that exists at \( T = 0 \).

To see the Landau level structure we thus need this smearing to be small on the scale of the spacing between the Landau levels.

i.e. need \( k_B T \ll \hbar \omega_0 \).
Using \( \omega_c = \frac{eH}{mc} \) and in the free electron mass one can compute

\[
\omega_c = 1.76 \times 10^7 \text{ sec}^{-1} \quad \text{for a } H = 1 \text{ tesla}
\]

\[
= 10^4 \text{ gauss}
\]

magnetic field.

1 tesla is a big field. In a laboratory set-up such as in BE one can buy a 1 tesla magnet. Larger field strengths require specialized facilities.

So for \( H = 1 \text{ tesla}, \) \( \frac{\hbar \omega_c}{k_B} = 1.34 \textbf{ } \text{K} \)

So in a 1 tesla field one needs to go well below 1 K to see Landau level structure.

In a 10 tesla field one needs to go well below 10 K. So quite low temperatures are needed.

There is a second condition: In solving Schrödinger's equation for the Landau levels, we ignored any sources of electron scattering (scattering
off phonons, plasmons, lattice impurities, etc.) If \( T \) is the scattering time, include such scattering generally leads, via the uncertainty principle, to a broadening of the energy levels of the eigenstates to a finite width \( \Delta E \approx \frac{1}{\hbar T} \)
So to see Landau level structure we need

\[ \frac{\kappa}{\hbar} \mu \Rightarrow \frac{E}{\mu} \ll \hbar \omega_c \]

\[ \Rightarrow \mu_c \gg 1 \]

Using \( \omega_c = 1.76 \times 10^9 \text{ sec}^{-1} \) in \( H = 1 \text{ Tesla} \)

and from resistivity measurements used to estimate \( \tau \) from Drude model we get

room temp \( \tau \sim 10^{-14} \text{ sec} \) \( \omega_c \tau \sim 0.00176 \)

77 K (liquid He) \( \tau \sim 10^{-13} \text{ sec} \) \( \omega_c \tau \sim 0.0176 \)

we again see that we will need very low temperatures (large \( \tau \)) to get \( \omega_c \tau \gg 1 \).

Landau level structure is typically only observable if one goes down to liquid Helium temperatures \( \sim 5 \text{ K} \).