An arbitrary weak but attractive interaction between two electrons excited above the filled Fermi surface leads to a bound state of the electrons with energy $E < 2E_F$. This then leads to an instability of the filled Fermi sphere to such pair bound pair formation, that completely changes the nature of the ground state of the $N$-electron system and leads to the phenomenon of superconductivity (BCS - Bardeen-Cooper-Schrieffer theory of superconductivity). The presence of the filled Fermi sphere is crucial to the effect - compare to two isolated particles in 3D where a bound state will not form unless the interaction exceeds a certain strength.

Consider a pair of electrons excited above the Fermi surface $E_F$. Assume that the ground state of this pair will have zero net momentum and zero net spin (singlet spin state). Since the interaction is attractive, most few electrons prefer to be near each other - most favorable wavefunction is spatially symmetric, so it must be antisymmetric in spin.
Let \( \vec{r}_1 \) and \( \vec{r}_2 \) be the positions of the two electrons. Assume that the two-particle wave function has the form:

\[
\psi(\vec{r}_1, \vec{r}_2) = \frac{1}{\sqrt{V}} \sum_k g_k e^{i \vec{k} \cdot \vec{r}_1} e^{-i \vec{k} \cdot \vec{r}_2} \quad (V \text{ is volume})
\]

Set \( \vec{k}_1 = -\vec{k}_2 \) so that total momentum of the pair is zero.

Since the electrons are above a filled Fermi sphere, we must have \( g_k = 0 \) for all \( |\vec{k}| < k_F \) since these states are already occupied.

If \( U(\vec{r}_1, \vec{r}_2) \) is the interaction between the two electrons, then the Schroedinger equation is

\[
-\frac{\hbar^2}{2m} \left[ \frac{\partial^2}{\partial \vec{r}_1^2} + \frac{\partial^2}{\partial \vec{r}_2^2} \right] \psi + U(\vec{r}_1, \vec{r}_2) \psi = E \psi
\]

Use Fourier transform, \( U(\vec{r}_1, \vec{r}_2) = \frac{1}{V} \sum_k U_k e^{i \vec{k} \cdot (\vec{r}_1 - \vec{r}_2)} \)

Plug into Schroedinger equation to get:

\[
\frac{1}{V} \sum_k \left[ \vec{k}^2 + \vec{k}_1^2 \right] \frac{\partial^2}{\partial \vec{r}_1^2} e^{i \vec{k} \cdot (\vec{r}_1 - \vec{r}_2)} + \frac{1}{V} \sum_k U_k e^{i \vec{k} \cdot (\vec{r}_1 - \vec{r}_2)} e^{i \vec{k} \cdot (\vec{r}_1 - \vec{r}_2)} = E \psi
\]

or

\[
\sum_k \left( \frac{\hbar^2 k^2}{m} \frac{\partial^2}{\partial \vec{r}_1^2} + \frac{V}{k^2} \sum_k U_{k-k'} g_{k'} - E g_k \right) e^{i \vec{k} \cdot (\vec{r}_1 - \vec{r}_2)} = 0
\]

where we made substitution \( \vec{r} = \vec{r}_1 - \vec{r}_2 \) in the potential term.
\[ \Rightarrow \frac{n^2 k^2}{m} g_k + \frac{1}{\sqrt{h}} \sum_{k'} U_{k-k'} g_{k'} = E g_k \]

Bethe - Goldstone equation

\[ g_k = 0 \quad \text{for} \quad |\mathbf{k}| < k_F \]

using \( E_k = \frac{n^2 k^2}{2m} \) we have

\[ (E - 2E_k) g_k = \frac{1}{\sqrt{h}} \sum_{k'} U_{k-k'} g_{k'} \]

This is very difficult to solve for a general \( U_{k-k'} \).

To simplify, we make a crude approximation:

max phonon energy \( \max \text{phonon energy} \)

\[ U_{k-k'} = \begin{cases} -U_0 & \text{if } E_k, E_{k'} \text{ within } \text{BZ of } \pm \varepsilon_p \\ 0 & \text{otherwise} \end{cases} \]

\[ \Rightarrow g_k = -\frac{U_0}{E - 2E_k} \left( \frac{1}{\sqrt{h}} \sum_{k'} g_{k'} \right) \]

where \( \sum \) means a sum over \( k' \) such that \( |\mathbf{k}'| > k_F \) and \( \frac{n^2 k^2}{2m} < \varepsilon_f + h \omega_s \)

Now sum both sides over \( k \)

\[ \left( \sum_k g_k \right) = -U_0 \left( \frac{1}{\sqrt{h}} \sum_{k'} g_{k'} \right) \left( \sum_k \frac{1}{E - 2E_k} \right) \]

cancel \( \sum_k g_k \) from both sides to get
\[ 1 = -U_0 \sqrt{\frac{Z}{k}} \frac{1}{E - 2\varepsilon_k} \]

\[ \Rightarrow 1 = -U_0 \int_{\varepsilon_F}^{\varepsilon_F + \hbar \omega_D} \frac{g(E)}{E - 2\varepsilon} \quad \text{where we assumed} \quad g(E) \text{ varied slowly from } \varepsilon_F \text{ to } \varepsilon_F + \hbar \omega_D \]

\[ \Rightarrow 1 = \frac{U_0 g(\varepsilon_F)}{Z} \ln \left( \frac{2\varepsilon_F - E + 2\hbar \omega_D}{2\varepsilon_F - E} \right) \]

\[ E = 2\varepsilon_F - \frac{2\hbar \omega_D}{Z g(\varepsilon_F) U_0} - 1 \]

For a weak potential, \( U_0 g(\varepsilon_F) \ll 1 \), we have

\[ E - 2\varepsilon_F = -2\hbar \omega_D e^{-2/g(\varepsilon_F) U_0} \]

Since the pair of electrons, in the absence of the attractive potential \( U \), would have a minimum energy of \( 2\varepsilon_F \), the binding energy of the pair is

\[ E' = 2\varepsilon_F - E = 2\hbar \omega_D e^{-2/g(\varepsilon_F) U_0} > 0 \]

\( \Rightarrow \) bound state \( E \) has a lower energy than \( 2\varepsilon_F \).
Note that the binding energy, \( E \sim -2g(E)u_0 \), is a non-analytic function of \( u_0 \), i.e., it cannot be expanded in powers of \( u_0 \). This means that we could never have gotten the result by using perturbation theory!

Conclusion: pairs of electrons at the Fermi surface \( E_F \) can lower their energy by binding together into such a "Cooper pair." \( \Rightarrow \) the filled Fermi sphere ceases to be stable to the formation of Cooper pairs and so can no longer be the true ground state. The new ground state was obtained by B.C.S. starting from the idea of Cooper pairs and became the basis for understanding superconductivity.

Back to the Cooper pair wavefunction:

\[ g \sim \frac{u_0 \times \text{constant}}{2E_F - E} \quad \text{only via } E_F \]

\( \Rightarrow \) solution is spherically symmetric \( \Rightarrow \) Cooper pairs bind in an S-wave state.

The maximum \( g \) occurs for the smallest \( E_F \), \( \Delta \) at \( k^2 = k_F^2 \). Using \( E = 2E_F - E' \), we get
\[ g_k \sim \frac{U_0}{2(E_k-E_F)+E'} \]

The distribution \( g_k \) as function of \( k \) has a width given by
\[ E_k - E_F \approx E' \ll \hbar W \]

Since only states within \( E' \) of \( E_F \) are important in making the bound state, and \( E' \ll \hbar W \), this suggests that the detailed structure of \( U_k - W \) is not so crucial and so our approximation might not be so terrible.

The size of a Cooper pair respective to \( \hbar \) can be estimated as follows:

\[ \delta_0 \sim \Delta R \sim \frac{1}{\Delta k} = \frac{1}{\frac{dE}{dE_k} \Delta E} = \frac{dE}{dE_k} \frac{1}{\Delta E} = \frac{\hbar V_F}{E'} \]

where we used that width in \( k \)-space was determined by \( E_k - E_F \approx E' \).

\[ \delta_0 \sim \frac{\hbar V_F}{E'} \Rightarrow \delta \quad \left( \frac{4 \pi e^2}{\hbar} \frac{3}{n} = \frac{1}{m} \right) \]

So the spatial extent of a Cooper pair is very much larger than the spacing between electrons.
What is the origin of the attractive e-e interaction that leads to Cooper pairing?
It is a time-delayed ionic screening effect!

\[ \text{ion} \rightarrow \text{ionic cloud} \]

The excess (+) charge where the ions have deformed them attract a new electron to the place where the first electron had been. This leads to an effective (but time-delayed) attractive correlation between the electrons.

Crucial to this picture is that the ions move much more slowly than the electrons, so the deformation that attracts the 2nd electron remains after the 1st electron has passed. That is how the two (-) electrons can still attract!